Stochastic properties of Radiation Fields

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Chapter 1

Intrinsic fluctuations

Irrespective of the type of information carrier, e.g. electromagnetic or particle radiation, the incoming radiation beam in astronomical measurements is subject to fluctuations which derive from the incoherent nature of the emission process in the information (=radiation) source. For a particle beam this is immediately obvious from the corpuscular character of the radiation, however for electromagnetic radiation the magnitude of the fluctuations depends on whether the wave character or the quantum character (i.e. photons) dominates. By employing Bose-Einstein statistics, the magnitude of these intrinsic fluctuations can be computed for the specific case of a blackbody radiation source which is incoherent or "chaotic" with respect to time. Photons are *bosons*, which are not subject to the Pauli exclusion principle, consequently many bosons may occupy the same quantum state.

1.1 Equilibrium distribution functions

The physical world is quantum-mechanical. One way of describing this, is to note that particles in a unit volume of space ('a cubic centimeter') are distributed in momentum-space in boxes with a size proportional to h^3, where h is Planck's constant. At each energy, there is a finite number

Z of boxes (where $Z \propto 4\pi p^2 dp$ with p the particle momentum).

1.1.1 Bose-Einstein statistics

Consider n_i particles, each with energy ϵ_i, and call the number of boxes available at that energy Z_i. *Bosons can share a box.* The number of ways $W(n_i)$ in which the n_i bosons can be distributed over the Z_i boxes is given by:

$$W(n_i) = \frac{(n_i + Z_i - 1)!}{n_i!(Z_i - 1)!} \tag{1.1}$$

(To understand this: the problem is equivalent to laying n_i particles and $Z_i - 1$ boundaries in a row. The number of permutations is $(n_i + Z_i - 1)!$, then note that the particles and boundaries can be interchanged.) Similarly, put n_j bosons in Z_j boxes, etc. The number of ways in which $N = \sum_{i=1}^{\infty} n_i$ bosons can be distributed over the boxes in momentum space thus is: $W = \Pi_{i=1}^{\infty} W(n_i)$.

The basic assumption of statistical physics is that the probability of a distribution is proportional to the number of ways in which this distribution can be obtained, i.e. to W. The physics behind this assumption is that collisions between the particles will continuously re-distribute the particles over the boxes. The most likely particle distribution is thus the one which can be reached in most different ways. This maximum in W can be found by determining the maximum of $\ln W$, i.e. by setting its derivative to zero:

$$\ln W = \sum_{i=1}^{\infty} \ln W(n_i) \Rightarrow \Delta \ln W = \sum_{i=1}^{\infty} \frac{\partial \ln W(n_i)}{\partial n_i} \Delta n_i = 0 \tag{1.2}$$

Consider now one term from the sum on the right-hand side. With Stirlings approximation $\ln x! \simeq x \ln x - x$ for large x one gets:

$$\begin{aligned} \ln W(n_i) &= (n_i + Z_i - 1)\ln(n_i + Z_i - 1) - (n_i + Z_i - 1) - n_i \ln n_i + n_i - \\ &\qquad - (Z_i - 1)\ln(Z_i - 1) + (Z_i - 1) = \\ &= (n_i + Z_i - 1)\ln(n_i + Z_i - 1) - n_i \ln n_i - (Z_i - 1)\ln(Z_i - 1) \end{aligned} \tag{1.3}$$

For a nearby number $n_i + \Delta n_i$, one has:

$$\begin{aligned} \Delta \ln W(n_i) &\equiv \ln W(n_i + \Delta n_i) - \ln W(n_i) \simeq \Delta n_i \frac{\partial \ln W(n_i)}{\partial n_i} \\ &= \Delta n_i \left[\ln(n_i + Z_i - 1) - \ln n_i\right] \end{aligned} \tag{1.4}$$

to first order in Δn_i.
The *most likely* particle distribution, representing the *equilibrium situation*, can now be obtained from setting:

$$\Delta \ln W \;=\; \sum_{i=1}^{\infty} \Delta n_i \left[\ln(\bar{n}_i + Z_i - 1) - \ln \bar{n}_i\right] \;=\; 0 \tag{1.5}$$

in which $\bar{n}_i$ represents the equilibrium (expectation, c.q. average) value of n_i. This notation is also used for other equilibrium parameters, e.g. the occupation number (see further on).

Since one considers a system in thermodynamic equilibrium, i.e. for which the number of particles $N = \sum n_i$ per unit volume and the energy $U = \sum n_i \epsilon_i$ per unit volume are constant, the variations in n_i must be such as to conserve N and U:

$$\Delta N = \sum_{i=1}^{\infty} \Delta n_i = 0 \tag{1.6}$$

and:

$$\Delta U = \sum_{i=1}^{\infty} \epsilon_i \Delta n_i = 0 \tag{1.7}$$

These restrictions imply that the following expression should also hold:

$$\Delta \ln W - \alpha \Delta N - \beta \Delta U \;=\; \sum_{i=1}^{\infty} \Delta n_i \left[\ln(\bar{n}_i + Z_i - 1) - \ln \bar{n}_i - \alpha - \beta \epsilon_i\right] \;=\; 0 \tag{1.8}$$

The sum in equation (1.8) is zero for arbitrary variations Δn_i, provided that for each i:

$$\ln(\bar{n}_i + Z_i - 1) - \ln \bar{n}_i - \alpha - \beta \epsilon_i = 0 \Rightarrow \frac{\bar{n}_i}{Z_i - 1} = \frac{1}{e^{\alpha + \beta \epsilon_i} - 1} \tag{1.9}$$

Since $Z_i \gg 1$, $\bar{n}_i/(Z_i - 1)$ can be replaced by $\bar{n}_i/Z_i$, which represents the average occupation at energy level ϵ_i (occupation number). The values of the coefficients α and β can in principle be obtained from the total number of particles and the total energy by substituting n_i in $N = \sum_{i=1}^{\infty} n_i$ and in $U = \sum_{i=1}^{\infty} n_i \epsilon_i$. However, their values can be more easily obtained by employing the following thermodynamical relations:

$$S \equiv k \ln W \Rightarrow \Delta \ln W \; = \; \frac{\Delta S}{k} \tag{1.10}$$

Substitution in expression (1.8) now yields:

$$\Delta S = k\alpha \Delta N + k\beta \Delta U \tag{1.11}$$

Furthermore, the fluctuation ΔU can be expressed as a function of the fluctuations ΔS, ΔV and ΔN according to the general thermodynamical relation:

$$\begin{aligned} \Delta U &= T\Delta S - p\Delta V + \zeta \Delta N, \qquad \text{for this case reducing to:} \\ \Delta U &= T\Delta S + \zeta \Delta N \end{aligned} \tag{1.12}$$

since the volume considered remains constant ($\Delta V = 0$). The parameter $\zeta \equiv -\alpha/\beta$ represents the thermodynamical potential per particle (the so-called *chemical potential*). The chemical potential can directly be obtained from the thermodynamic potential $G = U - TS + pV$, also referred to as the free enthalpy or Gibbs free energy of the system concerned, through $\zeta = G/N$.
Comparison with (1.11) results in $\alpha = -\zeta/(kT)$ and $\beta = 1/(kT)$. Substituting these values for α and β in equation (1.9) and, since $Z_i \gg 1$, replacing $\bar{n}_i/(Z_i - 1)$ by the average occupation number $\bar{n}_k = \bar{n}_i/Z_i$ (i.e. the average occupation at energy level $\epsilon_k \equiv \epsilon_i$) yields:

$$\bar{n}_k = \frac{1}{e^{\frac{\epsilon_k - \zeta}{kT}} - 1} \tag{1.13}$$

which is the *Bose Einstein equilibrium distribution.*
Note: To discriminate between the actual number of particles n_i at energy level ϵ_i with available levels Z_i and the associated occupation

number, no new variable has been introduced but rather a change in suffix from i to k. This implicitly means that in the relevant formulae $\epsilon_i \equiv \epsilon_k$ and $Z_i \equiv Z_k$ can be freely interchanged depending on whether the *actual* number or the *occupation* number is being treated.
End of Note.

In the case of a Boson gas comprising photons, the number of photons need not be conserved: e.g. an atom can absorb a photon and jump from orbit 1 to orbit 3, and then emit 2 photons by returning via orbit 2. Thus, the Lagrange condition given in equation (1.6) does not apply to photons, hence the photon distribution is obtained by dropping α in equation (1.9):

$$\frac{\bar{n}_i}{Z_i - 1} = \bar{n}_{\nu_k} = \frac{1}{e^{\frac{h\nu_k}{kT}} - 1} \tag{1.14}$$

which is the *Planck equilibrium distribution.*
$\bar{n}_{\nu_k}$ now represents the average occupation number of photons at frequency ν_k. Note that photons do not collide directly with one another, but reach equilibrium only via interaction with atoms.

1.1.2 Fermi Dirac statistics

The same treatment as in the previous section can be applied to fermions. However, in this case the *particles are not allowed to share a box*, and the number of ways $W(n_i)$ in which n_i particles can be distributed over Z_i boxes with energies ϵ_i is given by:

$$W(n_i) = \frac{Z_i!}{n_i!(Z_i - n_i)!} \tag{1.15}$$

The difference in $\ln W(n_i)$ between nearby numbers is:

$$\ln W(n_i + \Delta n_i) - \ln W(n_i) = -\Delta n_i \left[\ln n_i - \ln(Z_i - n_i)\right] \tag{1.16}$$

to first order in Δn_i. This leads to the following equilibrium distribution function:

$$\frac{\bar{n}_i}{Z_i} = \bar{n}_k = \frac{1}{e^{\frac{\epsilon_k - \zeta}{kT}} + 1} \tag{1.17}$$

which is the *Fermi Dirac equilibrium distribution.*

Chapter 2

Fluctuation around equilibrium

Assessment of fluctuations in the equilibrium state can be tackled from two different perspectives:

- A macroscopic fluctuation analysis of the fundamental thermodynamic quantities like temperature, pressure, volume and entropy
- A microscopic approach involving a statistical treatment of the particle population distribution functions.

Both approaches are covered in the following paragraphs to maximize the understanding of the underlying physics.

2.1 Fluctuations of the thermodynamic quantities T, S, p and V

In this section, the mean square fluctuations of the fundamental macroscopic thermodynamic quantities will be derived pertaining to any small part of a *closed system (cs)*, i.e. a system thermally isolated from its environment. The system can be regarded as a *medium* in which any small

part represents a *body* that can freely exchange heat and work with the medium. Owing to the large size of the medium relative to the body concerned, the temperature T and the pressure p of the medium can be taken as constant.
Let S_{cs} be the total entropy of the closed system (medium + body). If the body is in *equilibrium* with the medium, $S_{cs} = f(U_{cs})$ is at its *maximum value* for a given value of the total energy U_{cs}, i.e.:

$$(S_{cs})_{max} = f_{eq}\,(U_{cs}) \tag{2.1}$$

However, if the body is *not in equilibrium* (notation *) with the surrounding medium, the total entropy of the closed system is *lower* relative to its maximum value $(S_{cs})_{max}$ by a certain amount ΔS_{cs} for the *same value* of the total energy U_{cs}:

$$S^*_{cs} = (S_{cs})_{max} - \Delta S_{cs} \tag{2.2}$$

This is schematically shown by segment AB in figure (2.1). The horizontal segment BC gives the change in the total energy when the body goes *reversibly* from a state of equilibrium with its surrounding (medium) to a non-equilibrium state that corresponds to position B in figure (2.1). Putting it differently, the segment BC represents the minimum work that needs to be done by some external source to bring the body *reversibly* from a state of equilibrium with the medium to a state corresponding to position B. One has to keep in mind, that during this process the body may exchange work and heat with the medium. As a result, the total change ΔU in the energy of the body associated with a certain change in its state comprises three parts: the work w done on the body by the external source, the work done by the medium and the heat gained from the medium. Since the temperature T_m and the pressure p_m of the medium may be taken as a constant, the work done by the medium on the body equals $p_m \Delta V_m$ and the heat loss to the body is given by $-T_m \Delta S_m$. Thus:

$$\Delta U = w + p_m \Delta V_m - T_m \Delta S_m \tag{2.3}$$

Since the total volume of the closed system remains constant, $\Delta V_m = -\Delta V_{body} \equiv -\Delta V$, moreover the law of increase of entropy gives $\Delta S_{body} +$

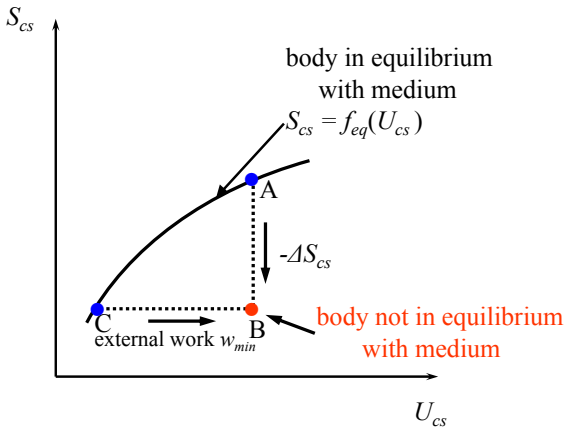

Figure 2.1: *The entropy of a closed thermodynamic system as a function of its total energy. The solid line represents the situation for full equilibrium. The vertical dashed line AB represents the drop in entropy $-\Delta S_{cs}$ when the body is not in equilibrium with the surrounding medium. The horizontal dashed line BC is the change in total energy when the body goes reversibly from the state of equilibrium with the medium to the non-equilibrium state corresponding to dot-B. Credit Landau and Lifshitz (1975).*

$\Delta S_m \geq 0$, hence $\Delta S_m \geq -\Delta S_{body} \equiv \Delta S$. This yields:

$$w \geq \Delta U - T_m \Delta S + p_m \Delta V \tag{2.4}$$

The equality occurs for a reversible process. The equilibrium (= mean) values of the temperature and the pressure of the body, $\bar{T}_{body}$ and $\bar{p}_{body}$ are by definition equal to T_m and the pressure p_m of the medium: $\bar{T}_{body} = T_m \equiv T$ and $\bar{p}_{body} = p_m \equiv p$. The suffixes of T and p have now been dropped since in what follows, it will be implicitly assumed that the coefficients of the fluctuation quantities always refer to their equilibrium (mean) values. Hence the minimum work by the external

source can also be expressed as:

$$w_{min} \;=\; \Delta(U - TS + pV) \;\equiv\; \Delta G \tag{2.5}$$

Equation (2.5) in fact shows that the work done by the external source is equal to the change in thermodynamic potential (or free enthalpy) ΔG of the body. Since the body considered constitutes only a very small part of the closed system, the processes involving it cause only a miniscule relative change in the total energy ΔU and the total entropy ΔS, hence one can immediately see from figure (2.1) that the following relation should hold:

$$\Delta S_{cs} \;=\; -\left(\frac{dS_{cs}}{dU_{cs}}\right) w_{min} \tag{2.6}$$

Moreover, the derivative $(\partial U_{cs}/\partial S_{cs})$ represents the equilibrium temperature T of the closed system, i.e. the temperature of the medium, thus:

$$\Delta S_{cs} \;=\; -\left(\frac{w_{min}}{T}\right) = -\left(\frac{\Delta U - T\,\Delta S + p\,\Delta V}{T}\right) \tag{2.7}$$

Expression (2.7) gives the amount by which the entropy of a closed system differs from its maximum value if the body is not in equilibrium with the medium. The values ΔU, ΔS and ΔV represent the differences between the energy, the entropy and the volume of the body and their values in the state of complete equilibrium.
According to equation (1.10), the probability to encounter a fluctuation in entropy ΔS at entropy value S is given by:

$$\ln W(S + \Delta S) - \ln W(S) \;=\; \Delta \ln W(S) \;=\; \frac{\Delta S}{k} \tag{2.8}$$

Substitution of relation (2.7) yields a fluctuation probability:

$$W \;=\; e^{-\left(\frac{\Delta U - T\,\Delta S + p\,\Delta V}{kT}\right)} \;=\; e^{-\left(\frac{\Delta G}{kT}\right)} \tag{2.9}$$

The exponential can be evaluated by writing the energy U as a function of the variables S and V. Taylor expansion of ΔU with inclusion of the

second order terms yields:

$$\Delta U = \left(\frac{\partial U}{\partial S}\right)_V \Delta S + \left(\frac{\partial U}{\partial V}\right)_S \Delta V + \\ + \frac{1}{2}\left[\left(\frac{\partial^2 U}{\partial S^2}\right)_V (\Delta S)^2 + 2\left(\frac{\partial U^2}{\partial S \partial V}\right)_{S,V} \Delta S \Delta V + \left(\frac{\partial^2 U}{\partial V^2}\right)_S (\Delta V)^2\right] \quad (2.10)$$

Substituting $(\partial U/\partial S)_V = T$ and $(\partial U/\partial V)_S = -p$, ΔG reduces to:

$$\begin{aligned} \Delta G &= \frac{1}{2}\left[\left(\frac{\partial^2 U}{\partial S^2}\right)_V (\Delta S)^2 + 2\left(\frac{\partial^2 U}{\partial S \partial V}\right)_{S,V} \Delta S \Delta V + \left(\frac{\partial^2 U}{\partial V^2}\right)_S (\Delta V)^2\right] \\ &= \frac{1}{2}\left[\Delta S \Delta\left(\frac{\partial U}{\partial S}\right)_V + \Delta V \Delta\left(\frac{\partial U}{\partial V}\right)_S\right] \\ &= \frac{1}{2}\left(\Delta S \Delta T - \Delta V \Delta p\right) \end{aligned} \quad (2.11)$$

This leads to a *fundamental result*, that expresses the fluctuation probability in terms of the fluctuations of all four macroscopic thermodynamic quantities ΔT, ΔS, Δp and ΔV:

$$W = e^{-\left(\frac{\Delta T \Delta S - \Delta p \Delta V}{2kT}\right)} \quad (2.12)$$

From this general expression the fluctuations of each of the four fundamental thermodynamic quantities can be derived. Each fluctuation quantity can be expressed as a function of two independent variables. Let's first take T and V as independent variables, i.e. $S = f(T,V)$ and $p = f(T,V)$:

$$\begin{aligned} \Delta S &= \left(\frac{\partial S}{\partial T}\right)_V \Delta T + \left(\frac{\partial S}{\partial V}\right)_T \Delta V = \frac{C_v}{T}\Delta T + \left(\frac{\partial p}{\partial T}\right)_V \Delta V \\ \Delta p &= \left(\frac{\partial p}{\partial T}\right)_V \Delta T + \left(\frac{\partial p}{\partial V}\right)_T \Delta V \end{aligned} \quad (2.13)$$

with C_v the specific heat under constant volume and with application of the thermodynamic Maxwell relation $(\partial S/\partial V)_T = (\partial p/\partial T)_V$.

Substitution of the above expressions in equation (2.12) yields:

$$W = e^{-\left[\frac{C_v}{2kT^2}(\Delta T)^2 - \frac{1}{2kT}\left(\frac{\partial p}{\partial V}\right)_T (\Delta V)^2\right]} = e^{-\left[\frac{C_v}{2kT^2}(\Delta T)^2\right]} . e^{-\left[-\frac{1}{2kT}\left(\frac{\partial p}{\partial V}\right)_T (\Delta V)^2\right]} \tag{2.14}$$

This result shows that the temperature and volume fluctuations are independent from each other ($\overline{\Delta T \Delta V} = 0$), and that both distributions are Gaussian with:

$$\begin{aligned} &\text{Mean:} && \overline{\Delta T} &&= 0 \\ &\text{Variance:} && \overline{(\Delta T)^2} &&= \frac{kT^2}{C_v} \qquad (2.15) \\ &\text{Mean:} && \overline{\Delta V} &&= 0 \\ &\text{Variance:} && \overline{(\Delta V)^2} &&= -kT \left(\frac{\partial V}{\partial p}\right)_T , \qquad \left(\frac{\partial V}{\partial p}\right)_T < 0 \,! \qquad (2.16) \end{aligned}$$

To assess the magnitude of the fluctuations Δp and ΔS, take now p and S as independent variables, i.e. $T = f(p, S)$ and $V = f(p, S)$:

$$\begin{aligned} \Delta T &= \left(\frac{\partial T}{\partial p}\right)_S \Delta p + \left(\frac{\partial T}{\partial S}\right)_p \Delta S = \left(\frac{\partial T}{\partial p}\right)_S \Delta p + \frac{T}{C_p} \Delta S \\ \Delta V &= \left(\frac{\partial V}{\partial p}\right)_S \Delta p + \left(\frac{\partial V}{\partial S}\right)_p \Delta S = \left(\frac{\partial V}{\partial p}\right)_S \Delta p + \left(\frac{\partial T}{\partial p}\right)_S \Delta S \qquad (2.17) \end{aligned}$$

with C_p the specific heat under constant pressure and with application of the thermodynamic Maxwell relation $(\partial V/\partial S)_p = (\partial T/\partial p)_S$.
Substitution of the above expressions in equation (2.12) yields:

$$W = e^{-\left[\frac{1}{2kC_p}(\Delta S)^2 - \frac{1}{2kT}\left(\frac{\partial V}{\partial p}\right)_S (\Delta p)^2\right]} = e^{-\left[\frac{1}{2kC_p}(\Delta S)^2\right]} . e^{-\left[-\frac{1}{2kT}\left(\frac{\partial V}{\partial p}\right)_S (\Delta p)^2\right]} \tag{2.18}$$

This result shows that the entropy and pressure fluctuations are independent from each other ($\overline{\Delta S \Delta p} = 0$), and that both distributions are Gaussian with:

$$\begin{aligned} &\text{Mean:} && \overline{\Delta S} &&= 0 \\ &\text{Variance:} && \overline{(\Delta S)^2} &&= kC_p \qquad (2.19) \\ &\text{Mean:} && \overline{\Delta p} &&= 0 \\ &\text{Variance:} && \overline{(\Delta p)^2} &&= -kT \left(\frac{\partial p}{\partial V}\right)_S , \qquad \left(\frac{\partial p}{\partial V}\right)_S < 0 \,! \qquad (2.20) \end{aligned}$$

2.2 Fluctuations in the equilibrium state

The mean square volume fluctuation given by expression (2.16) can be used to also derive the fluctuation in the number of particles in a fixed volume in the body.
Consider some part of the body that contains N particles. The volume fluctuation *per particle* follows from dividing both sides by N^2:

$$\overline{(\Delta(V/N))^2} = -\frac{kT}{N^2}\left(\frac{\partial V}{\partial p}\right)_T \tag{2.21}$$

For a *fixed* volume, $\Delta(V/N) = -(V/N^2)\Delta N$, and equation (2.21) can be written as:

$$\overline{(\Delta N)^2} = -kT\left(\frac{N}{V}\right)^2\left(\frac{\partial V}{\partial p}\right)_T \tag{2.22}$$

This expression can be reworked by implementing the following notions. The partial derivative $(\partial V/\partial p)_T$ was taken under the implicit condition that N is constant, hence:

$$\begin{aligned} -kT\left(\frac{N}{V}\right)^2\left(\frac{\partial V}{\partial p}\right)_T &= -kT\left(\frac{N}{V}\right)^2\left(\frac{\partial V}{\partial p}\right)_{T,N} = \\ &= kTN\left[\frac{\partial}{\partial p}\left(\frac{N}{V}\right)\right]_{T,N} \end{aligned} \tag{2.23}$$

Furthermore, the number of particles $N = Vf(p,T)$, so $(N/V) = f(p,T)$ and consequently it is immaterial whether (N/V) is differentiated at constant N or constant V, thus the variance for the fluctuation distribution in the number of particles becomes:

$$\begin{aligned} \overline{(\Delta N)^2} &= kTN\left[\frac{\partial}{\partial p}\left(\frac{N}{V}\right)\right]_{T,N} = kTN\left[\frac{\partial}{\partial p}\left(\frac{N}{V}\right)\right]_{T,V} = \\ &= kT\left(\frac{N}{V}\right)\left(\frac{\partial N}{\partial p}\right)_{T,V} = kT\left(\frac{\partial N}{\partial p}\right)_{T,V}\left(\frac{\partial p}{\partial \zeta}\right)_{T,V} = \\ &= = kT\left(\frac{\partial N}{\partial \zeta}\right)_{T,V} \end{aligned} \tag{2.24}$$

In the above sequence the equality $(N/V) = (\partial p/\partial \zeta)_{T,V}$ was utilized. This equality follows directly from the expression for the differential of the free energy ΔF by inclusion of a term that takes the fluctuation of the number of particles into account: $\Delta F = \Delta(U - TS) = -S\Delta T - p\Delta V + \zeta \Delta N$. Considering a fixed temperature ($\Delta T = 0$) and a fixed volume ($\Delta V = 0$) and transforming $\zeta \Delta N$ in the equivalent $N\Delta\zeta$ through $\zeta \Delta N = \Delta(\zeta N) - N\Delta\zeta = \Delta G - N\Delta\zeta$, one arrives at $\Delta(F - G) = -\Delta(pV) = -N\Delta\zeta$. With $\Delta V = 0$ this results in $(\partial p/\partial \zeta)_{T,V} = N/V$.

Note: Expression (2.24) can also be derived directly from the distribution function of the micro-canonical ensemble describing the equilibrium state of a closed system. For completeness this derivation is also presented here.
Consider a thermodynamically closed system that constitutes a large ensemble of subsystems. Each of these subsystems in turn comprises a large number of particles N. The probability $\mathbf{p}(U_k) \equiv \mathbf{p}_k$ that such a subsystem possesses a microscopically defined quantum state with total energy U_k is given by the Gibbs distribution:

$$\mathbf{p}_k = A \cdot e^{-\left(\frac{U_k}{kT}\right)} \quad \text{with} \quad A^{-1} = \sum_k e^{-\left(\frac{U_k}{kT}\right)} \tag{2.25}$$

with $\mathbf{p}_k$ the probability that the subsystem concerned possesses a total energy U_k associated with its microscopically defined quantum state. The value of the coeffient A follows from the normalization requirement that the integral probability taken over all energy levels U_k should be equal to unity. This expression implicitly assumes however that the number of particles N in the subsystem is constant. If the fluctuation in the number of particles is to be taken into account, as it should be in this case, the micro-canonical distribution needs to be extended to both energy U_k *and* particle number N described by the bivariate Gibbs distribution function $\mathbf{p}(N, U_k) \equiv \mathbf{p}_{N,k}$:

$$\mathbf{p}_{N,k} = Ae^{-\left(\frac{U_{N,k} - \zeta N}{kT}\right)} \qquad \text{with the normalization:}$$

$$A^{-1} \quad = \quad \sum_N \sum_k e^{-\left(\frac{U_{N,k}\,-\,\zeta N}{kT}\right)} \;=\; \sum_N e^{\left(\frac{\zeta N}{kT}\right)} \sum_k e^{-\left(\frac{U_{N,k}}{kT}\right)} \tag{2.26}$$

with $\zeta = G/N$ the chemical potential as before. Note that the quantized total energy levels $U_{N,k}$ of the subsystems are now also a function of the actual number of particles N contained in each specific subsystem involved.
The fluctuation in the number of particles can now be obtained by computing the variance $\overline{(\Delta N)^2} = \overline{N^2} - \bar{N}^2$ by applying the probability distribution function (2.26):

$$\bar{N} = \frac{\sum\limits_N N e^{\left(\frac{\zeta N}{kT}\right)} \sum\limits_k e^{-\left(\frac{U_{N,\,k}}{kT}\right)}}{\sum\limits_N e^{\left(\frac{\zeta N}{kT}\right)} \sum\limits_k e^{-\left(\frac{U_{N,\,k}}{kT}\right)}} \quad \text{and} \quad \overline{N^2} = \frac{\sum\limits_N N^2 e^{\left(\frac{\zeta N}{kT}\right)} \sum\limits_k e^{-\left(\frac{U_{N,\,k}}{kT}\right)}}{\sum\limits_N e^{\left(\frac{\zeta N}{kT}\right)} \sum\limits_k e^{-\left(\frac{U_{N,\,k}}{kT}\right)}} \tag{2.27}$$

Introducing the variable $J = \sum\limits_N e^{(\zeta N)/(kT)} \sum\limits_k e^{-(U_{N,\,k})/(kT)}$, one can write for (2.27):

$$\bar{N}^2 \quad = \quad \frac{1}{J^2}\left(kT\frac{\partial J}{\partial \zeta}\right)^2 \quad \text{and} \quad \overline{N^2} \;=\; \frac{(kT)^2}{J}\frac{\partial^2 J}{\partial \zeta^2}$$

and after substitution:

$$\overline{(\Delta N)^2} \quad = \quad (kT)^2 \frac{\partial}{\partial \zeta}\left(\frac{1}{J}\frac{\partial J}{\partial \zeta}\right) \;=\; kT\left(\frac{\partial \bar{N}}{\partial \zeta}\right) \tag{2.28}$$

which concurs with the result that was obtained previously in (2.24).
End of Note

If N is distributed according to a Bose-Einstein equilibrium distribution, substituting $N = \bar{n}_k$ yields the variance in the occupation number due to the intrinsic fluctuations:

$$\overline{(\Delta n_k)^2} \quad = \quad kT\left(\frac{\partial \bar{n}_k}{\partial \zeta}\right)_{T,V}$$

$$
\begin{aligned}
&= kT\left(\frac{\partial}{\partial\zeta}\right)_{T,V}\left(\frac{1}{e^{\frac{\epsilon_k-\zeta}{kT}}-1}\right)\\
&= \left(\frac{1}{e^{\frac{\epsilon_k-\zeta}{kT}}-1}\right)\cdot\left(1+\frac{1}{e^{\frac{\epsilon_k-\zeta}{kT}}-1}\right), \quad \text{hence:}\\
\overline{(\Delta n_k)^2} &= \bar{n}_k(1+\bar{n}_k) \qquad (2.29)
\end{aligned}
$$

Equivalently, when considering a photon gas with the Planck equilibrium distribution the intrinsic fluctuations amount to a variance:

$$\overline{(\Delta n_{\nu_k})^2} = \bar{n}_{\nu_k}(1+\bar{n}_{\nu_k}) \qquad (2.30)$$

Similarly, applying expression (2.24) to a Fermi-Dirac equilibrium distribution yields a variance for the fermion distribution of:

$$\overline{(\Delta n_k)^2} = \bar{n}_k(1-\bar{n}_k) \qquad (2.31)$$

A different, and more direct, approach to assess the fluctuation in equilibrium distributions is to extend the statistical treatment used for establishing the particle population distribution functions that resulted in the Bose and Fermi distributions.
The approach involves a look into the number of ways in which $n_i+\Delta n_i$ particles can be distributed, as compared to that for n_i particles:

$$\ln W(n_i+\Delta n_i) = \ln W(n_i) + \Delta n_i\frac{\partial \ln W(n_i)}{\partial n_i} + \frac{(\Delta n_i)^2}{2}\frac{\partial^2 \ln W(n_i)}{\partial n_i^2} \qquad (2.32)$$

where now the 2nd order term has been included.
In equilibrium the term in equation (2.32), proportional to Δn_i, is zero. Consequently, equation (2.32) can be written as:

$$W(n_i+\Delta n_i) = W(n_i)\cdot e^{-\frac{W''(n_i)}{2}(\Delta n_i)^2} \text{ with } W''(n_i) \equiv -\frac{\partial^2 \ln W(n_i)}{\partial n_i^2} \qquad (2.33)$$

In other words, the probability of a deviation Δn_i drops exponentially with the square of Δn_i, i.e. the probability of Δn_i is given by a Gaussian! The average value for $(\Delta n_i)^2$ is found by integrating over all values of Δn_i:

$$\overline{(\Delta n_i)^2} = \frac{\int_{-\infty}^{\infty} (\Delta n_i)^2 W(n_i) \cdot e^{-\frac{W''(n_i)}{2}(\Delta n_i)^2} d\Delta n_i}{\int_{-\infty}^{\infty} W(n_i) \cdot e^{-\frac{W''(n_i)}{2}(\Delta n_i)^2} d\Delta n_i} = \frac{1}{W''(n_i)} \quad (2.34)$$

Note: $W(n_i)$ and $W''(n_i)$ do not depend on Δn_i, so that they can be considered as constants in the integrations. The maximum negative deviation has $\Delta n_i = -n_i$, and the maximum positive deviation $\Delta n_i = N - n_i$, consequently the integrals should formally be evaluated between these values; however, for large Δn_i the integrand drops rapidly to zero, and so the integrals can be extended to the full range $-\infty$ to $+\infty$, without compromising the result.
End of Note.

Computing the second derivative of $\ln W(n_i)$ and changing sign yields the variance:

$$\overline{(\Delta n_i)^2} = [W''(n_i)]^{-1} = \frac{\bar{n}_i(\bar{n}_i + Z_i - 1)}{Z_i - 1} = \bar{n}_i \left[1 + \frac{1}{e^{\frac{\epsilon_i - \zeta}{kT}} - 1}\right] \quad (2.35)$$

As argued before, in case of a photon gas ζ needs to be dropped. Hence, for photons:

$$\overline{(\Delta n_i)^2} = \bar{n}_i(1 + \bar{n}_{\nu_k}) = \bar{n}_i \left[1 + \frac{1}{e^{\frac{h\nu_k}{kT}} - 1}\right] \quad (2.36)$$

Hence, the fluctuation in the average occupation number $\overline{(\Delta n_{\nu_k})^2}$ for a *photon gas* follows from:

$$\overline{(\Delta n_{\nu_k})^2} = \frac{\overline{(\Delta n_i)^2}}{Z_i} = \bar{n}_{\nu_k}(1 + \bar{n}_{\nu_k}) \quad (2.37)$$

For the case of a *Fermi-Dirac equilibrium distribution*, the average value

of the square of the deviation can be derived in a way similar to the one deployed for the Bose Einstein distribution:

$$\overline{(\Delta n_i)^2} = \frac{\bar{n}_i(Z_i - \bar{n}_i)}{Z_i} = \bar{n}_i(1 - \bar{n}_k) = \bar{n}_i \left[1 - \frac{1}{e^{\frac{\epsilon_k - \zeta}{kT}} + 1}\right] \tag{2.38}$$

The fluctuation in the average occupation number $\overline{(\Delta n_k)^2}$ now follows from:

$$\overline{(\Delta n_k)^2} = \frac{\overline{(\Delta n_i)^2}}{Z_i} = \bar{n}_k(1 - \bar{n}_k) \tag{2.39}$$

2.3 Fluctuations in a blackbody radiation field

The average volume density of photons in a blackbody Bose gas with frequencies between ν_k and $\nu_k + d\nu_k$ follows from $\bar{N}(\nu_k)d\nu_k = g(\nu_k)\bar{n}_{\nu_k}d\nu_k$, in which $g(\nu_k)$ represents the volume density of quantum states per unit frequency at ν_k. Since the stochastic variables n_{ν_k} are independent, the Bose-fluctuations $\overline{(\Delta N)^2}(\nu_k)$ in photon density per unit frequency can be written as (omitting the suffix k):

$$\overline{(\Delta N)^2}(\nu) = \bar{N}(\nu)\left(1 + \frac{1}{\exp(h\nu/kT) - 1}\right) \tag{2.40}$$

where $\bar{N}(\nu)$ follows from the specific energy density $\bar{\rho}(\nu) = \rho(\nu)^{equilibrium}$ given by:

$$\bar{\rho}(\nu)d\nu = \frac{8\pi h}{c^3}\frac{\nu^3}{\exp\left(\frac{h\nu}{kT}\right) - 1}d\nu \tag{2.41}$$

through the relation $\bar{N}(\nu) = \bar{\rho}(\nu)/h\nu$.

If a detection element, e.g. an antenna, is placed within a blackbody radiation field, for example inside a vacuum enclosure at temperature T, the incident photon flux is given by $\bar{n}(\nu) = \frac{1}{2}\frac{c}{4\pi}\bar{N}(\nu)A_e\Omega$. The factor $\frac{1}{2}$ refers to *one component of polarisation*, A_e is the effective area of the detection element and Ω constitutes the solid angle subtended by the detector beam viewing the radiation field. If radiation illuminates an

extended surface (A_e) with various directions of the wave vector, i.e. an omnidirectional blackbody radiation field, coherence theory states that spatial coherence is limited to $A_e\Omega \approx \lambda^2$, the so-called *extent (etendue) of coherence*.[1] Substituting $\bar{N}(\nu)$, the expression for the specific photon flux $\bar{n}(\nu)$ (in photons s^{-1} Hz^{-1}) becomes:

$$\bar{n}(\nu) = \frac{1}{\exp(h\nu/kT) - 1} \tag{2.42}$$

$$\overline{(\Delta n)^2}(\nu) = \bar{n}_\nu \left(1 + \frac{1}{\exp(h\nu/kT) - 1}\right) \tag{2.43}$$

In the extreme case that $h\nu \gg kT$, the second term becomes much smaller than 1, hence:

$$\overline{(\Delta n)^2}(\nu) = \bar{n}(\nu) \tag{2.44}$$

This is the well-known expression for Poissonian noise in a sample containing $\bar{n}(\nu)$ photons. This condition is called the *quantum limit* of the fluctuations and it represents the minimum value of intrinsic noise present in any radiation beam. Obviously, this always holds for corpuscular radiation (cosmic-rays) and neutrinos, since the wave character is not an issue.

For a photon energy $h\nu \ll kT$, the noise is normally expressed in terms of the average radiation power $\bar{P}(\nu)$ (e.g. in Watt Hz^{-1}) by writing $\bar{P}(\nu) = (h\nu)\bar{n}(\nu)$ and $\overline{(\Delta P)^2}(\nu) = (h\nu)^2\overline{(\Delta n)^2}(\nu)$:

$$\overline{(\Delta P)^2}(\nu) = \bar{P}(\nu)\left(h\nu + \frac{h\nu}{\exp(h\nu/kT) - 1}\right) = \bar{P}(\nu)(h\nu + \bar{P}(\nu)) \tag{2.45}$$

Taking the limit $h\nu \ll kT$:

$$\overline{(\Delta P)^2}(\nu) = \bar{P}^2(\nu) \tag{2.46}$$

$$\text{and:} \quad \bar{P}(\nu) = kT \tag{2.47}$$

which is the expression for the classical thermal noise power per unit frequency bandwidth. This limit is called the *thermal limit*.

[1] This relation is the same as that governing the size $\theta = \lambda/D$ of a diffraction limited beam ($\Omega \approx \theta^2$) for an aperture with diameter D: $A_e \approx D^2$.

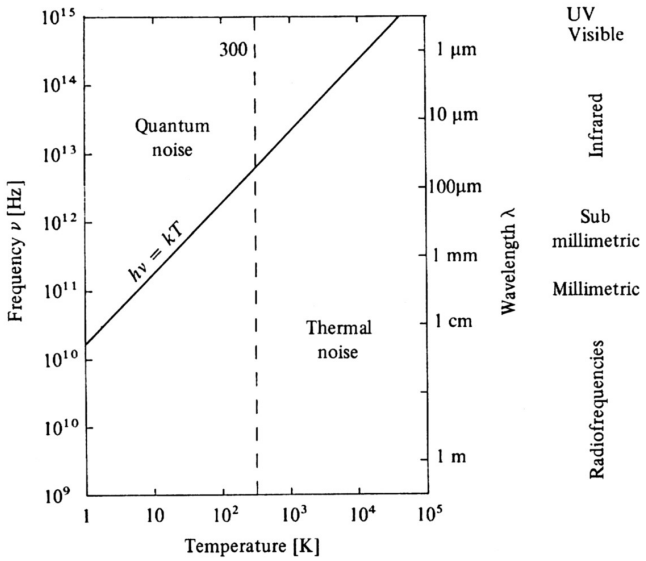

Figure 2.2: *Frequency as a function of temperature: division between thermal and quantum noise. Credit Lena (1988).*

The transition between noise in the quantum limit to the thermal limit occurs at $h\nu \approx kT$. At room temperature, $T \approx 300$ K, this corresponds to a frequency $\nu \approx 6$ THz, or a wavelength $\lambda \approx 50$ μm. The relation $\nu = kT/h$ as a function of temperature T is displayed in figure 2.2. It is clear from this diagram that radio observations are always dominated by the wave character of the incoming beam and are therefore carried out in the thermal limit. As a result, the treatment of noise in radio observations differs drastically from that of measurements at shorter wavelengths. Specifically at submillimetric and infrared wavelengths quantum limited observation is vigorously pursued but this remains still difficult.

The fluctuations in average power $\bar{P}(\nu)$, given in equation 2.46, for the thermal limit can be interpreted in such a way that whenever wave

packet interference becomes important, the interference will cause the fluctuations to become of the same magnitude as the signal. The low frequency fluctuations can be thought of as caused by the random phase differences and beats of the wavefields, as described in the following paragraph.

Note of Caution: The above expression for the fluctuations in a blackbody photon gas applies only to the interior of a blackbody in which the receiving element is submerged, i.e. a blackbody cavity or a thermal bath, where the condition $\lambda^2 = c^2/\nu^2 = A_e\Omega$ *is satisfied. If this condition is not fulfilled, then even in the limit* $h\nu \ll kT$ *quantum noise may dominate. For example, if a star, whose spectrum resembles a blackbody at temperature* T*, is observed at frequency* ν*, such that* $h\nu \ll kT$*, thermal noise ought to dominate. The star may however be so distant that the radiation is effectively unidirectional and hence,* $A_e\Omega \ll \lambda^2$*. The photons will consequently arrive well separated in time and quantum noise evidently dominates.*

Chapter 3

Stochastic description in the wave limit

3.1 Wave packets, random superposition

In astrophysics, many sources of electromagnetic (EM) radiation have a thermal origin. A beam of thermal radiation will ordinarily comprise a myriad of randomly overlapping wave groups or *wave packets*. Examples of such wave packets are shown in figure (3.1). They arise from two different types of quasi-monochromatic sources:
- Gaussian shaped spectral lines which emerge when several line broadening mechanisms contribute to the line formation (*viz.* the central limit theorem).
- Spectral lines with a Lorentz profile, characteristic for the natural line broadening associated with the intrinsic time spread of the radiative atomic transitions involved that are governed by the uncertainty relation of Heisenberg $\sigma_\epsilon \sigma_t = \hbar/2$.
The characteristic length, τ_c, of these wave packets in the time domain follows from the Fourier transform of the spectral frequency distribution. A Gaussian spectral line, centered at frequency ν_0 with a full-width half-maximum (FWHM) $\Delta\nu$, corresponds to a Gaussian shaped wave packet

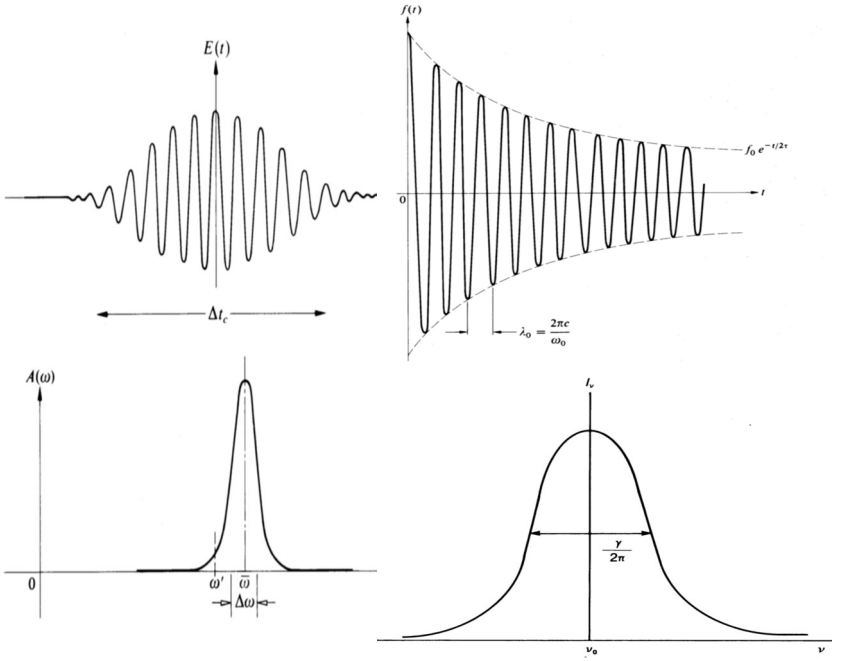

Figure 3.1: *Examples of wave packets and their Fourier (⇔) transforms. Left: Gaussian shaped wave packet ⇔ Gaussian spectral line. Right: Exponentially damped harmonic oscillator ⇔ Lorentzian spectral line. Credit Hecht (1987).*

with a characteristic width $\tau_c \simeq (1.5\Delta\nu)^{-1}$. A Lorentzian profile ($\nu_0$, FWHM $\Delta\nu$) derives from an exponentially damped harmonic oscillator with a $1/e$-value $\tau_c = (\pi\Delta\nu)^{-1}$. The characteristic time τ_c is commonly referred to as the *coherence time*, it represents the typical time scale over which the phase of the EM-wave can be predicted with reasonable accuracy at a *given location in space*. For atomic transitions in the optical $\tau_c \simeq 10^{-9}$s. Figure (3.2) shows a linearly polarised quasi-monochromatic signal comprising a random superposition of individual wavepackets. This wave signal fluctuates both in amplitude and in frequency, the latter

characterised by a typical bandwidth $\Delta\nu$ around an average frequency $\bar{\nu}$. The frequency stability of such a quasi-monochromatic wave is defined by $\bar{\nu}/\Delta\nu$.

The linearly polarised signal displayed in figure (3.2) can mathematically be expressed by the real function:

$$E(t) = E_0(t)\cos(2\pi\bar{\nu}t + \phi(t)) \tag{3.1}$$

The amplitude $E_0(t)$ of the quasi-monochromatic wave is a *wide-sense stationary* Gaussian random time function of zero mean. Moreover the stochastic process is assumed to be *mean- and correlation-ergodic*, i.e. for an arbitrary real stochastic variable $X(t)$ its *expectation value* at time t, $\mathbf{E}\{X(t)\}$, can be interchanged with its time average:

$$\bar{X} \;=\; \lim_{T\to\infty}\frac{1}{T}\int_{-\frac{1}{2}T}^{+\frac{1}{2}T} X(t)\,dt = \mathbf{E}\{X(t)\} \tag{3.2}$$

Furthermore, the expectation value at time t of the autocorrelation $R_X(\tau)$ of $X(t)$ can be interchanged with its time average:

$$\begin{aligned} R_X(\tau) \;&=\; \overline{X(t)\cdot X(t+\tau)} \;=\; \lim_{T\to\infty}\frac{1}{T}\int_{-\frac{1}{2}T}^{+\frac{1}{2}T} X(t)\cdot X(t+\tau)\,dt \;= \\ &=\; \mathbf{E}\{X(t)\cdot X(t+\tau)\} \end{aligned} \tag{3.3}$$

The frequency is randomly varying around an average frequency $\bar{\nu}$, the instantaneous frequency $\nu(t)$ follows from the time derivative of the argument of the cosine term according to:

$$\nu(t) \;=\; \frac{1}{2\pi}\frac{d}{dt}\left(2\pi\bar{\nu}t \,+\, \phi(t)\right) \;=\; \bar{\nu} \;+\; \frac{1}{2\pi}\frac{d\phi(t)}{dt} \tag{3.4}$$

As can be seen from equation (3.4), the time variable phase factor $\phi(t)$ fully accomodates the frequency bandwidth $\Delta\nu$ of the stochastic wave signal.

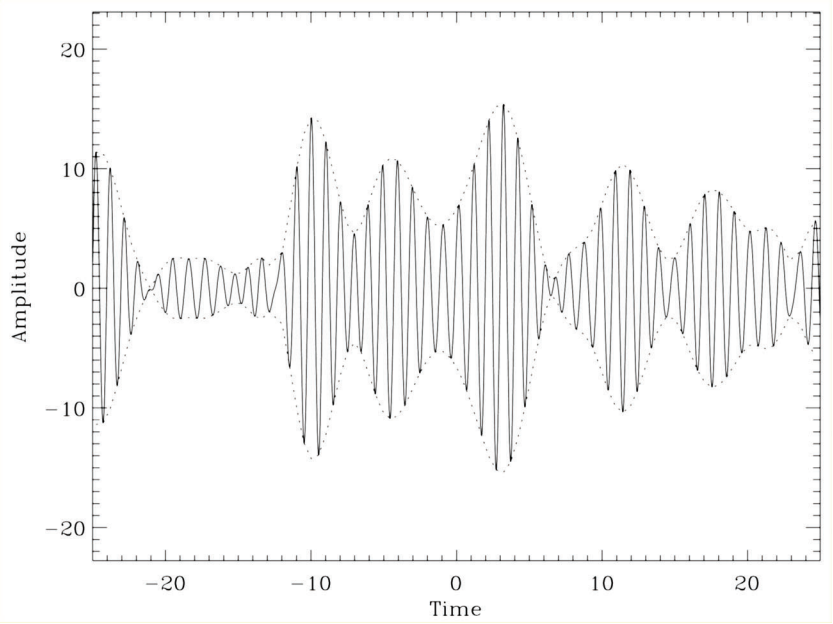

Figure 3.2: *A linearly polarised quasi-monochromatic wave.*

The above mathematical description suffices for a linearly polarised signal, however in case of a thermal radiator a particular *polarisation direction is only very short-lived*, i.e. only during the *coherence time* τ_c *of an emitted wave packet.* This can be understood by considering the emission process involved.
Thermal emission consists of an extremely large number of radiative transitions, generated by randomly oriented atomic emitters. Each atom radiates a polarised wave train for roughly 10^{-8} or 10^{-9} seconds in the case of optical light depending on the natural line width $\Delta\nu$ of the transition. In the case of molecular vibrational or rotational transitions (radio and far infrared) the timescales are substantially longer. Considering

a certain wave propagation direction $\vec{k}$, individual atomic (molecular) emissions along that direction will combine to generate a polarised wave, which however will not persist for more than the typical coherence time τ_c of a wave packet, i.e. in the optical $10^{-8} - 10^{-9}$ seconds. New wave trains are continually emitted and as a result the magnitude and the polarisation direction of the electric vector $\vec{E}(t)$ changes in a completely random manner on a typical time scale equal to the coherence time τ_c. If these changes occur at a rate of 10^8 to 10^9 per second, any persistent polarisation state is undiscernable. Thermal radiation is therefore designated as *natural or unpolarised light*, although the latter qualification is somewhat confusing since in actuality the light comprises a rapid succession of different polarisation states.

3.2 The analytic signal

The *rapid random fluctuations* in the electric vector $\vec{E}(t)$ of a thermal radiation field can be handled mathematically in a *scalar approach* by using a *complex analytic representation* of the quasi-monochromatic wave field.
Consider the time-varying electric field $E(t)$ of equation (3.1). Along with $E(t)$ one may consider a complex function:

$$\tilde{E}(t) = E(t) + iF_{Hi}(t) \tag{3.5}$$

in which:

$$F_{Hi}(t) \;=\; -\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{E(t')dt'}{t'-t} \tag{3.6}$$

is the Hilbert transform of $E(t)$. This integral can be interpreted as a convolution of $E(t)$ with $(\pi t)^{-1}$:

$$F_{Hi}(t) \;=\; E(t) \;*\; \frac{1}{\pi t} \tag{3.7}$$

Applying the convolution theorem and considering the Fourier transform of $(\pi t)^{-1} \;\Leftrightarrow\; i\,\mathrm{sgn}(\nu)$, the Hilbert transform can be regarded as a *special*

filter that leaves the amplitude of the spectral components unimpaired, but alters their phases by $\pi/2$, positively or negatively depending on the sign of ν. A consequence of this is, that Hilbert transforms of even functions are odd and those of odd functions even.
The complex function $\tilde{E}(t)$ of equation (3.5) is known as the *analytic signal*, the Hilbert transform is referred to as the *quadrature function* of $E(t)$. For example, the quadrature function of $\cos t$ is $\sin t$, the analytic signal is therefore $\exp(it)$.
Analytic functions are useful to describe quasi-monochromatic wave phenomena, where one deals with *modulated carrier* signals. The analytic signal contains *no negative frequency* components, it is obtainable from $E(t)$ by suppressing the negative frequencies and doubling the result. For example $\cos 2\pi\nu_0 t$ contains frequency components at ν_0 and $-\nu_0$:

$$\cos 2\pi\nu_0 t = \frac{e^{2\pi i\nu_0 t} + e^{-2\pi i\nu_0 t}}{2}, \tag{3.8}$$

the analytic signal follows from suppression of $e^{-2\pi i\nu_0 t}$ and multiplication by 2.

Problem 1: Show with the aid of a Fourier transform that applying the above operation to $E(t)$ results in the analytic signal $\tilde{E}(t)$!

If $E(t)$ is a Gaussian process, its Hilbert transform (linear) is also a Gaussian process, moreover the autocorrelation functions are equal and the values of $E(t)$ and its Hilbert transform are *uncorrelated* at the same instant t. The analytic signal comprises a harmonic oscillation at an average frequency $\bar{\nu}$ modulated by a slowly varying envelope:

$$\tilde{E}(t) = \tilde{E}_0(t) \cdot e^{i(2\pi\bar{\nu}t)} \tag{3.9}$$

The complex amplitude (envelope function) $\tilde{E}_0(t)$ can be expressed as:

$$\tilde{E}_0(t) = |\tilde{E}_0(t)| \cdot e^{i\phi(t)} \tag{3.10}$$

This envelope function is also referred to as the *phasor* of the analytic signal, $|\tilde{E}_0(t)|$ represents the instantaneous amplitude of $\tilde{E}(t)$ and $\phi(t)$ the time variable phase. The time rate of change of $\phi(t)$, i.e. $\left(\frac{1}{2\pi}\right)\frac{d\phi(t)}{dt}$,

represents the instantaneous frequency shift $\Delta\nu(t)$ of the analytic signal relative to the average carrier frequency $\bar{\nu}$.

In the next section, a quantitative fluctuation analysis of a thermal wave field will now be deployed with the aid of the expression for the analytic signal:

$$\tilde{E}(t) = |\tilde{E}_0(t)| \cdot e^{i[2\pi\bar{\nu}t+\phi(t)]} \tag{3.11}$$

3.3 Fluctuation analysis of a thermal wave field

The ideal monochromatic plane wave

Let's first consider the phasor expression for a, hypothetical, perfectly monochromatic plane wave and its physical interpretation.
The frequency bandwidth $\Delta\nu$ then reduces to a delta function: $\delta(\nu - \bar{\nu})$. In the time domain this is represented as an infinitely long wave train (i.e. the Fourier transform of a δ-function). If this wave train were to be resolved in two orthogonal polarisation components perpendicular to the direction of propagation, they in turn must have the same frequency, be infinite in extent and are therefore *mutually coherent*. In other words *an idealised monochromatic plane wave is always polarised*. Expressing this in terms of the phasor $\tilde{E}_0(t)$ of a linearly polarised plane wave, this results in:

$$\tilde{E}_0(t) = |\tilde{E}_0(t)| \cdot e^{i\phi(t)} = |\tilde{E}_0| \cdot e^{i\phi_0} \tag{3.12}$$

i.e. both the amplitude $|\tilde{E}_0|$ and phase ϕ_0 of the phasor are constant in time.

Polarized thermal light

Consider a radiation field emitted by a thermal source, that is subsequently passed by a polarization analyzer with its polarization direction along the X-axis in a plane perpendicular to the wave propagation vector $\vec{k}$. The time wave form in the *X - t plane* can now be regarded as the sum of a great many independent contributions of the x-components of

the fields generated by the individual atoms (molecules). The random fluctuations of $\vec{E}(t)$ along the x-axis, $\vec{E}_X(t)$, can now be described by a fluctuation analysis of the phasor $\tilde{E}_{X0}(t)$, i.e. of its magnitude $|\tilde{E}_{X0}(t)|$ and its phase $\phi(t)$. For time scales short compared to the coherence time (τ_c), $\tilde{E}_{X0}(t)$ will remain almost constant in time, in the case of optical light for $\tau_c \approx 10^{-8}$ s this still comprises several million harmonic oscillations of the electric vector $\vec{E}(t)$ ($\bar{\nu} \approx$ few 10^{14} Hz). For time scales $\tau \gg \tau_c$, $|\tilde{E}_{X0}(t)|$ and $\phi(t)$ will vary randomly, this is pictorially shown in figure 3.3. The coloured phasors signify the built up of $\tilde{E}_{X0}(t)$ by a great many independent atomic contributions that are fully uncorrelated. Mathematically these fluctuations can be described by regarding the real and imaginary parts of $\tilde{E}_{X0}(t)$, $Re(\tilde{E}_{X0}(t))$ and $Im(\tilde{E}_{X0}(t))$, as uncorrelated Gaussian stochastic variables with equal standard deviation that vary rapidly and randomly and are mutually incoherent.

The joint (bivariate) probability density distribution is then given by:

$$\begin{aligned} \mathbf{p}_{biv}\left(Re\tilde{E}_{X0}(t), Im\tilde{E}_{X0}(t)\right)\, dRe\tilde{E}_{X0}(t)\, dIm\tilde{E}_{X0}(t) \;&= \\ = \frac{1}{2\pi\sigma^2}\, e^{-\frac{Re^2\tilde{E}_{X0}(t)+Im^2\tilde{E}_{X0}(t)}{2\sigma^2}}\, dRe\tilde{E}_{X0}(t)\, dIm\tilde{E}_{X0}(t) \end{aligned} \tag{3.13}$$

Furthermore, the following relations hold:

$$|\tilde{E}_{X0}(t)|^2 = |\tilde{E}_X(t)|^2 = Re^2\tilde{E}_{X0}(t) + Im^2\tilde{E}_{X0}(t) \tag{3.14}$$

$$\begin{aligned} \phi(t) &= \arg(\tilde{E}_{X0}(t)) = \arg(\tilde{E}_X(t)) - 2\pi\bar{\nu}t \;= \\ &= \arctan\frac{Im\tilde{E}_{X0}(t)}{Re\tilde{E}_{X0}(t)} \end{aligned} \tag{3.15}$$

Changing to polar coordinates the bivariate probability density distribution for $|\tilde{E}_{X0}(t)|$ and $\phi(t)$ can be obtained:

$$\begin{aligned} \mathbf{p}_{biv}\left(|\tilde{E}_{X0}(t)|, \phi(t)\right)\, d\,|\tilde{E}_{X0}(t)|\; d\phi(t) \;&= \\ = \frac{|\tilde{E}_{X0}(t)|}{2\pi\sigma^2}\, e^{-\frac{|\tilde{E}_{X0}(t)|^2}{2\sigma^2}}\, d\,|\tilde{E}_{X0}(t)|\; d\phi(t) \end{aligned} \tag{3.16}$$

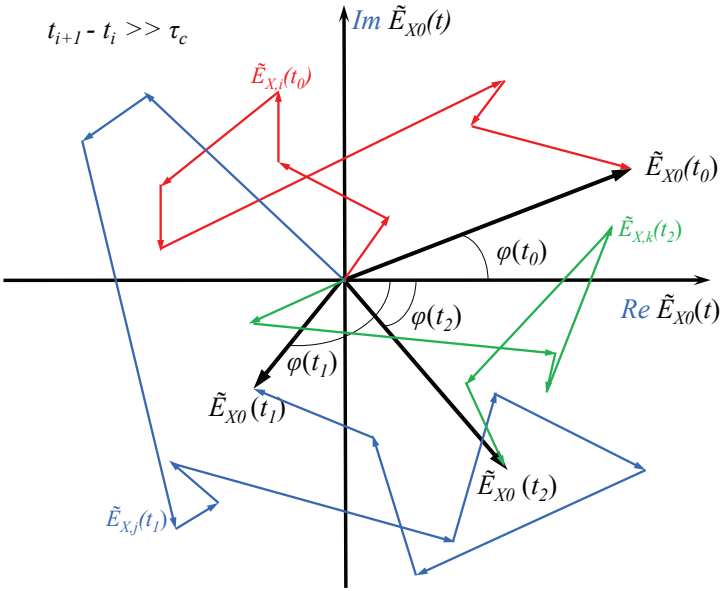

Figure 3.3: *Black: Random orientation of the complex envelopes (phasors) of polarized thermal light at a fixed point in space at three different times t_0, t_1 and t_2 separated by time intervals larger than the coherence time τ_c. Colors: signifying the great many independent complex atomic (molecular) phasors.*

Integration over $|\tilde{E}_{X0}(t)|$ yields:

$$\mathbf{p}\left(\phi(t)\right) = \frac{1}{2\pi} \tag{3.17}$$

i.e. all phase angles $\phi(t)$ are equally probable, which is obviously to be expected for randomly fluctuating phase angles.

Integration over all phase angles $\phi(t)$ yields the amplitude distribution $|\tilde{E}_{X0}(t)|$ for the X-component of the electric field vector of the

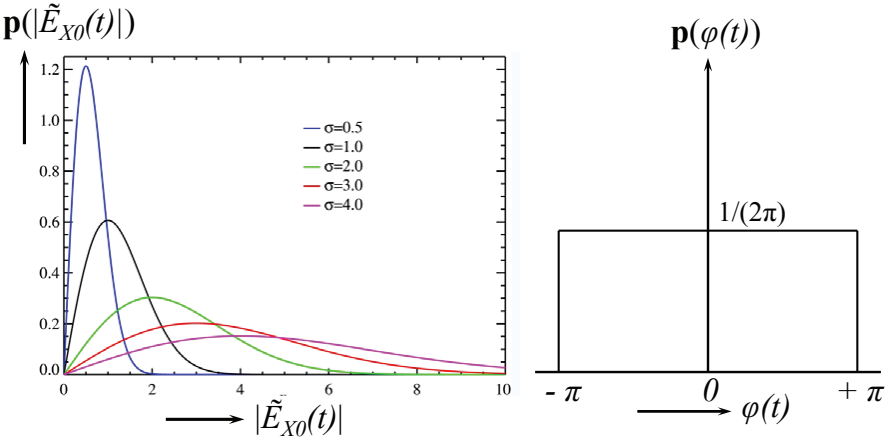

Figure 3.4: *The distribution functions* $\mathbf{p}\left(|\tilde{E}_{X0}(t)|\right)$ *and* $\mathbf{p}\left(\phi(t)\right)$.

thermal source:

$$\mathbf{p}_R\left(|\tilde{E}_{X0}(t)|\right) = \frac{|\tilde{E}_{X0}(t)|}{\sigma^2} \cdot e^{-\frac{|\tilde{E}_{X0}(t)|^2}{2\sigma^2}} \tag{3.18}$$

This is a so-called *Rayleigh distribution.* $\mathbf{p}_R\left(|\tilde{E}_{X0}(t)|\right)$ and $\mathbf{p}\left(\phi(t)\right)$ are displayed in figure 3.4.

Problem: 2 Show that the most probable value of $|\tilde{E}_{X0}(t)|$ equals σ

and the average value for the amplitude equals $\sigma\sqrt{(\pi/2)}$. ◁

The intensity (irradiance) of a traveling plane EM-wave

The energy streaming through space in the form of an electromagnetic wave is shared between the constituent electric and magnetic fields.
The energy density of an electrostatic field (e.g. between plates of a capacitor) $\rho_{\vec{E}} = \epsilon_r\epsilon_0|\vec{E}|^2/2$ (dimension Joule/m^3), with $|\vec{E}|$ the magnitude of the electric vector (dimension V/m) and ϵ_0 the vacuum permittivity ($8.8543 \cdot 10^{-12}$ Asec/Vm). Similarly, the energy density of a magnetic field (e.g. within a toroid) equals $\rho_{\vec{B}} = |\vec{B}|^2/(2\mu_r\mu_0)$ (dimension Joule/m^3), with $|\vec{B}|$ the magnitude of the magnetic vector (dimension Tesla = Vsec/m^2) and μ_0 the vacuum permeability ($4\pi \cdot 10^{-7}$ Vsec/Am).
The wave equation for a **plane electromagnetic wave** traveling along the x-direction in vacuum is given by:

$$\frac{\partial^2 E(x,t)}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 E(x,t)}{\partial t^2} \quad \text{and} \quad \frac{\partial^2 B(x,t)}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 B(x,t)}{\partial t^2} \qquad (3.19)$$

for the electric field wave and the magnetic field wave respectively. The magnetic field wave travels in a plane perpendicular to the electric field, both the electric field and the magnetic field directions are perpendicular to the direction of propagation (x). The plane wave solution can be expressed by a harmonic function, using a complex scalar representation:

$$\tilde{E}(x,t) = E_0 \cdot e^{i\cdot 2\pi(\nu t - x/\lambda)} \quad \text{and} \quad \tilde{B}(x,t) = B_0 \cdot e^{i\cdot 2\pi(\nu t - x/\lambda)} \qquad (3.20)$$

Consistency with Maxwell's equations requires that for the EM-wave holds $\rho_{\vec{E}} = \rho_{\vec{B}}$. Hence, from the above, we have $B_0 = E_0/c$.
The flow of electromagnetic energy through space associated with the traveling EM-wave is represented by the Poynting vector $\vec{S} = (1/\mu_0)\vec{E}\,\mathbf{x}\,\vec{B}$, a vector product that symbolizes the direction and magnitude of the energy transport per unit time across a unit area (e.g. in units Watt m^{-2}). The vector magnitude $|\vec{S}| = |\tilde{E}||\tilde{B}|(\sin\phi)/\mu_0$ equals $|\tilde{E}||\tilde{B}|/\mu_0$, since the magnetic field is perpendicular to the electric field ($\phi = \pi/2$). Representing the *actual* wave signal by taking the real part of expressions

(3.20) we get:

$$|\vec{S}| = E_0 B_0 \cos^2 2\pi(\nu t - x/\lambda) = \epsilon_0 c E_0^2 \cos^2 2\pi(\nu t - x/\lambda) = (\epsilon_0/\mu_0)^{\frac{1}{2}} E_0^2 \cos^2 2\pi(\nu t - x/\lambda) \quad (3.21)$$

The *average* power flux density for an *ideal monochromatic* plane wave, $\overline{I(t)}$ equals $\overline{|\vec{S}(t)|}$:

$$\overline{I(t)} = (\epsilon_0/\mu_0)^{\frac{1}{2}} E_0^2 \overline{\cos^2 2\pi(\nu t - x/\lambda)} = (\epsilon_0/\mu_0)^{\frac{1}{2}} \frac{E_0^2}{2} = 2.6544 \cdot 10^{-3} \frac{E_0^2}{2} \quad (3.22)$$

expressed in Watt/m^2 for E_0 in Volts/meter.
A perfectly monochromatic plane wave is represented in the time domain by an infinitely long wave train and is by definition *fully polarised*. A *quasi-monochromatic* radiation field from a thermal source can be described by a complex expression for the electric field $\tilde{E}(t)$, comprising a harmonic oscillation at an average frequency $\bar{\nu}$ modulated by a slowly varying envelope, accomodated by the phasor $\tilde{E}_0(t)$, i.e. $\tilde{E}(t) = \tilde{E}_0(t) \cdot e^{i(2\pi\bar{\nu}t)}$. The average power flux density for this wave then follows from the expectation value of the product $\tilde{E}(t) \cdot \tilde{E}^*(t)$:

$$\overline{I(t)} = \mathbf{E}\{I(t)\} = (\epsilon_0/\mu_0)^{\frac{1}{2}} \mathbf{E}\left\{\tilde{E}(t) \cdot \tilde{E}^*(t)\right\} = 2.6544 \cdot 10^{-3} \mathbf{E}\left\{|\tilde{E}_0(t)|^2\right\} \quad (3.23)$$

Since we are primarily concerned with *relative power flux densities generated by these traveling waves within the same medium*, we can disregard in what follows multiplication with the numerical constant in expression (3.23), since this (deterministic) quantity is only of relevance for assessing the *absolute numerical value* of the power flux density and bears no influence on the description of the stochastic nature of the signals. In practical computations, this constant should of course be applied!

The probability density distribution of the instantaneous intensity of *polarized* thermal light

From the distribution derived for $|\tilde{E}_{X0}(t)|$ in a previous paragraph, the probability density of the instantaneous intensity (or irradiance) $I_X(t)$ for a linearly polarized (along the X-axis) thermal radiation field can be readily derived.
We can now set the following equalities:

$$I_X(t) = \tilde{E}_X(t) \cdot \tilde{E}_X^*(t) \;=\; |\tilde{E}_X(t)|^2 \;=\; |\tilde{E}_{X0}(t)|^2 \tag{3.24}$$

Transformation of variables in equation (3.18) yields:

$$\mathbf{p}(I_X)\; dI_X = (\bar{I}_X)^{-1} \cdot e^{-I_X/\bar{I}_X}\; dI_X \tag{3.25}$$

with $\bar{I}_X = \mathbf{E}\left\{|\tilde{E}_{X0}(t)|^2\right\} = 2\sigma^2$.
This is an exponential probability density distribution that has the important property that its standard deviation σ_{I_X} is equal to its mean $\bar{I}_X$ $(= 2\sigma^2)$.

Problem 3: Prove that the variance of this intensity probability distribution is given by:

$$\overline{(\Delta I_X)^2} = \bar{I}_X^2 \tag{3.26}$$

◁

This result, which is now obtained formally from the bivariate Gaussian distributed stochastic process with zero-mean for the harmonic wave components, is the same as the fluctuation in the average radiation power per unit frequency bandwidth (Watt Hz^{-1}) for *one component of polarization* that was derived earlier for a blackbody radiation field, i.e. $\overline{(\Delta P)^2}(\nu) = \bar{P}^2(\nu)$. This should evidently hold.

The probability density distribution of the instantaneous intensity of *unpolarized* thermal light

Light from a thermal source can be designated as unpolarized if it fullfils two conditions. First: if passed by a polarization analyzer in a plane perpendicular to the propagation vector $\vec{k}$, the intensity should be independent of the rotational orientation of the analyzer. Second: two orthogonal field components $\tilde{E}_X(t)$ and $\tilde{E}_Y(t)$ should have the property that $\tilde{E}_X(t)\tilde{E}_Y^*(t+\tau)| = 0$ for all rotational orientations of the X-Y coordinate axes and for all delays τ.
As demonstrated in the previous paragraphs, $\tilde{E}_X(t)$ and $\tilde{E}_Y(t)$ can be handled as complex Gaussian random processes and since they are uncorrelated for all relative time delays τ they are statistically independent. The instanteneous intensity of the unpolarized wave field is then defined by:

$$\begin{aligned} I(t) &= |\tilde{E}_X(t)|^2 + |\tilde{E}_Y(t)|^2 \\ &= |\tilde{E}_{X0}(t)|^2 + |\tilde{E}_{Y0}(t)|^2 \\ &= I_X(t) + I_Y(t) \end{aligned} \tag{3.27}$$

From the definition of unpolarized light $I_X(t)$ and $I_X(t)$ have equal means $\bar{I}_X = \bar{I}_Y = \frac{1}{2}\bar{I}$ and are independent statistical processes.
The probability density function of the total instanteneous intensity $I(t)$ follows from the density function of the sum of $\mathbf{p}(I_X)$ and $\mathbf{p}(I_Y)$:

$$\mathbf{p}(I_X) = 2(\bar{I})^{-1} \cdot e^{-2I_X/\bar{I}} \tag{3.28}$$

$$\mathbf{p}(I_Y) = 2(\bar{I})^{-1} \cdot e^{-2I_Y/\bar{I}} \tag{3.29}$$

The probability density function of this sum is the convolution of $\mathbf{p}(I_X)$ and $\mathbf{p}(I_Y)$:

$$\mathbf{p}(I) = \mathbf{p}(I_X) * \mathbf{p}(I_Y) \tag{3.30}$$

Hence:

$$\mathbf{p}(I) = 4(\bar{I})^{-2} \int_0^I e^{-2I'/\bar{I}} e^{-2(I-I')/\bar{I}} dI' = 4(\bar{I})^{-2} I e^{-2I/\bar{I}} \tag{3.31}$$

This density function is plotted in figure 3.5.
From this distribution it is clear that unpolarized thermal light has considerably less probability of having very small values of the instantaneous intensity than polarized thermal light. Moreover, the ratio of standard deviation σ to mean value $\bar{I}$ has reduced from unity in the case of a polarized thermal wave to $\sqrt{1/2}$ for an unpolarized thermal wave field. In practice a radiation beam is generally neither completely polarised nor completely unpolarised, both cases are extremes. More often the electric vector $\vec{E}(t)$ varies in a way that is neither totally regular nor totally irregular, the radiation should be regarded as partially polarised.

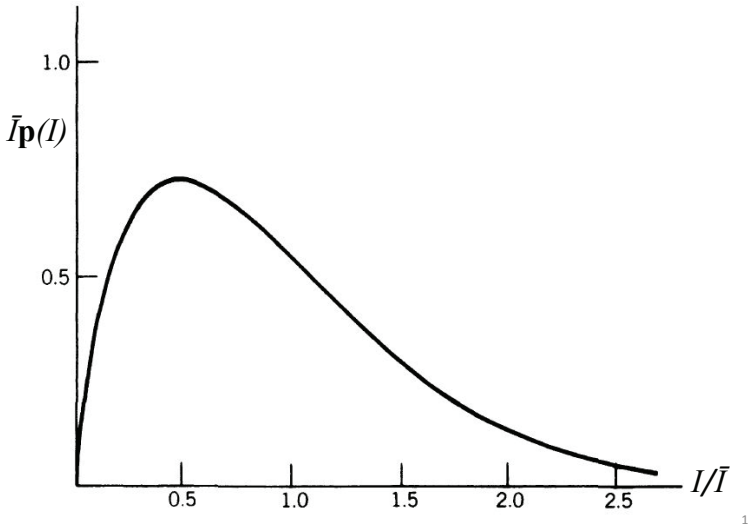

Figure 3.5: *Probability density function of the instantaneous intensity of an unpolarized thermal wave field. Credit Goodman (2000).*

An useful way to describe this behaviour is to envision it as the result of the superposition of specific amounts of natural and polarised light. If a quantitative assessment of the degree of polarisation is required, a measurement of all four Stokes parameters is required. In radio astronomy this is often done due to the intrinsic sensitivity of the receiver front-end for a particular direction of polarisation.

Chapter 4

Stochastic description in the quantum limit

4.1 The unfiltered Poisson process

In the quantum limit no coherence effects occur and the radiation field fluctuations can be described by photon statistics only. Consider an incident radiation beam (wide-sense stationary, ergodic) with an average flux of λ photons (or particles or neutrinos) per second. The generation of photons at random times t_i can be described by a staircase function, with discontinuities at time locations t_i (see figure 4.1):

$$Z(t) = \sum_i U(t - t_i),\ U(t) = \text{unit-step function} \tag{4.1}$$

$$U(t) = \begin{cases} 1 & \text{for } t \geq 0 \\ 0 & \text{for } t < 0 \end{cases}$$

The photon *flow rate* [number of photons per second] follows from time

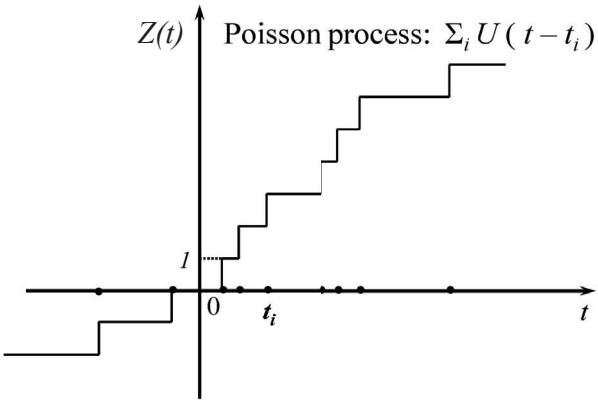

Figure 4.1: *Staircase function describing a Poisson process.*

differentiation of the stochastic variable $Z(t)$:

$$X(t) = \frac{dZ(t)}{dt} = \sum_i \delta(t - t_i) \tag{4.2}$$

and represents a train of Dirac impulses at random time locations t_i. At a constant photon rate, $X(t)$ is a wide sense stationary (WSS) stochastic signal with a *time independent* average $\overline{X(t)} = \lambda$ photons per second, λ is the *rate parameter* characteristic for the process under consideration. We can now express the stochastic process $Z(t)$, displayed in figure (4.1), in the following way:

$$Z(t) = \int_0^t X(t')dt' = \int_0^t \sum_i \delta(t' - t_i)dt' = k(0,t) \tag{4.3}$$

in which $k(t_1, t_2)$ represents the *number* of photons in a time period (t_1, t_2) of length $t = t_2 - t_1$. This number $k(t_1, t_2)$ is a Poisson distributed random variable (RV) with parameter λt, i.e. $Z(t)$ expresses an

unfiltered Poisson process:

$$\mathbf{p}\{k, \lambda t\} = \frac{(\lambda t)^k}{k!} e^{-\lambda t}, \quad \text{with } \lambda \text{ the rate parameter (see above)} \quad (4.4)$$

Note: For Poisson distributed RVs hold that if two time periods (t_1, t_2) and (t_3, t_4) are considered that are non-overlapping, then the RVs $k(t_1, t_2)$ and $k(t_3, t_4)$ are independent.
From expression (4.4) we can construct a Poissonian probability density function featuring a continuous random variable (κ):

$$\mathbf{p}(\kappa, \lambda t) = \sum_{k=0}^{\infty} \mathbf{p}(k, \lambda t)\delta(\kappa - k) \quad (4.5)$$

The average value of κ and of κ^2 for assessment of the fluctuation magnitude follow from:

$$\mathbf{E}\{\kappa\} = \int_{-\infty}^{+\infty} \kappa\ \mathbf{p}(\kappa, \lambda t) d\kappa = \lambda t \quad (4.6)$$

$$\mathbf{E}\{\kappa^2\} = \int_{-\infty}^{+\infty} \kappa^2\ \mathbf{p}(\kappa, \lambda t) d\kappa = (\lambda t)^2 + \lambda t \quad (4.7)$$

The average value for $\kappa = \lambda t$ in equation (4.6) is of course as expected; the first term of equation (4.7) is the square of the average and its second term represents the variance. Since the variance of the fluctuations associated with the flow of the photons equals λt, the standard deviation becomes $\sqrt{\lambda t}$, i.e. the 'strength' of the noise in the photon flow. The relative fluctuation or signal to noise ratio (SNR) is then:

$$SNR = \frac{\lambda t}{\sqrt{\lambda t}} = \sqrt{\lambda t} \quad (4.8)$$

Consequently, the larger λt, the smaller the relative shot noise in the photon flow. With very small λ we apparently need a long filter time to suppress this shot noise.

To determine the autocorrelation function $R_Z(t_1, t_2)$ of the Poisson process $Z(t)$ let us first consider $t_2 \geq t_1$. The variables $k(0, t_1)$ and $k(t_1, t_2)$, referring to adjacent but non-overlapping time periods, are then independent Poisson variables with parameters λt_1 and $\lambda(t_2 - t_1)$ respectively. Thus we have:

$$\begin{aligned}
\mathbf{E}\{k(0,t_1)k(t_1,t_2)\} &= \mathbf{E}\{k(0,t_1)\}\mathbf{E}\{k(t_1,t_2)\} = \\
&= \lambda^2 t_1(t_2 - t_1), \Rightarrow \\
k(t_1,t_2) &= k(0,t_2) - k(0,t_1) = Z(t_2) - Z(t_1), \Rightarrow \\
\mathbf{E}\left\{Z(t_1)\left[Z(t_2) - Z(t_1)\right]\right\} &= R_Z(t_1,t_2) - \mathbf{E}\{Z^2(t_1)\} \Rightarrow \\
R_Z(t_1,t_2) &= \lambda^2 t_1(t_2 - t_1) + \lambda^2 t_1^2 + \lambda t_1 = \\
&= \lambda^2 t_1 t_2 + \lambda t_1 \qquad (4.9) \\
\text{If } t_2 < t_1 \Rightarrow R_Z(t_1,t_2) &= \lambda^2 t_1 t_2 + \lambda t_2 \qquad (4.10)
\end{aligned}$$

Introducing the autocovariance $C_Z(t_1, t_2)$ of $Z(t)$ we can write:

$$\begin{aligned}
R_Z(t_1,t_2) &= \lambda^2 t_1 t_2 + C_Z(t_1,t_2) = \\
&= \lambda^2 t_1 t_2 + \lambda t_1 U(t_2 - t_1) + \lambda t_2 U(t_1 - t_2) \qquad (4.11)
\end{aligned}$$

Regarding the stochastic variable $X(t)$, the time derivative of $Z(t)$ and representing the train of Dirac impulses at random time locations, we have the time independent average value $\mathbf{E}\{X(t)\} = \lambda =$ the rate parameter.
The autocorrelation function follows from successive partial differentiation of the autocorrelation of $Z(t)$ with respect to t_1 and t_2, thus:

$$R_X(t_1,t_2) = \frac{\partial^2 R_Z(t_1,t_2)}{\partial t_1 \partial t_2} = \lambda + \delta(t_2 - t_1) \qquad (4.12)$$

Designating the time difference $(t_2 - t_1) = \tau$, we arrive at the general expression for the autocorrelation of a train of unit-value Dirac impulses at random time positions (WSS ergodic signal):

$$R_X(\tau) = \lambda^2 + \lambda\delta(\tau) \qquad (4.13)$$

The second term in equation (4.13) represents the covariance $C(\tau)$ of $X(t)$, which equals in this case the variance $C(0)$ since it is zero for

every value of τ except for $\tau = 0$. This is of course evident, since the Dirac impulses are randomly distributed in time and are thus mutually completely uncorrelated.

4.2 Frequency filtered Poisson process

By applying the Wiener Khinchin theorem to $R_X(\tau)$ we can compute the power spectral density:

$$R_X(\tau) \Leftrightarrow S_{d_X}(\nu) = \int_{-\infty}^{+\infty} R_X(\tau) e^{-2\pi j\nu\tau} d\tau = \lambda^2\delta(\nu) + \lambda \qquad (4.14)$$

which is inconsistent with physical reality since it implies an infinitely high power signal. In practice there is always a frequency cut-off at say ν_c, owing to some (high frequency) filtering process. We might perceive this as follows. The photon detection process involves conversion to charge carriers that are subsequently fed into a filter network, e.g. a first order RC filter. The RC-network acts on each individual charge impuls q (δ-function) with a current response function $h(t)$. If we now assume for convenience that each single photon generates a charge carrier (detection efficiency=1), implying also an average charge carrier rate λ, the resulting photo-current follows from a convolution of the Dirac δ-function train $X(t)$ with $h(t)$:

$$\frac{I(t)}{q} \;=\; X(t) \rightarrow h(t) \rightarrow Y(t) \qquad (4.15)$$

with $h(t)$ the filter circuit *impulse* current response function (*Note*: $h(t) = 0$ for $t < 0$ and is a normalized function: $\int_0^\infty h(t)dt = 1$).

Hence we have:

$$\begin{aligned} Y(t) &= h(t) * X(t) = \int_0^\infty \sum_k \delta(t' - t_k) h(t - t') dt' = \\ &= \sum_k h(t - t_k) = \overline{Y} + \Delta Y(t) \end{aligned} \qquad (4.16)$$

Owing to the high carrier density in the charge flow, there will be a large degree of overlap between subsequent responses. This will result in a total current $I(t)$ that shows a Gaussian (normal) distribution around a mean value $\overline{I}$. For the expectation value of $Y(t)$ we thus find

$$\begin{aligned} \overline{Y} &= \mathbf{E}\{Y(t)\} = \mathbf{E}\{\int_0^\infty X(t-t')h(t')dt'\} \\ &= \int_0^\infty \mathbf{E}\{X(t-t')\}\, h(t')dt' = \lambda \int_0^\infty h(t')dt' = \lambda\, H(0) \end{aligned} \quad (4.17)$$

where for the last transition we have used:

$$H(2\pi j\nu) \equiv \int_0^\infty h(t')e^{-2\pi j\nu t'}dt' \Rightarrow H(0) = \int_0^\infty h(t')dt' \quad (4.18)$$

In the Fourier domain we write for the *current* power spectral density:

$$\begin{aligned} S_{d_Y}(\nu) &= |H(2\pi j\nu)|^2\, S_{d_X}(\nu) \\ &= \lambda^2\, |H(2\pi j\nu)|^2\, \delta(\nu) + \lambda\, |H(2\pi j\nu)|^2 = \\ &= \lambda^2\, H^2(0) + \lambda\, |H(2\pi j\nu)|^2 \end{aligned} \quad (4.19)$$

Evidently the power is now finite, as it should be. We obtain the autocorrelation by taking the Fourier transform of the *current* power spectral density $S_{d_Y}(\nu)$:

$$R_Y(\tau) = \lambda^2\, H^2(0) + \lambda\, [h(\tau) * h(\tau)] \quad (4.20)$$

where the first term on the right hand side gives the *mean* charge response of the linear dynamic system, and the second term represents the noise. Taking the autocovariance at $\tau = 0$ we obtain the variance of the noise signal:

$$C_Y(0) = \lambda \int_{-\infty}^{+\infty} h^2(t)dt = \lambda \int_{-\infty}^{+\infty} |H(2\pi j\nu)|^2 d\nu =$$

$$= \quad 2\lambda \int_0^{+\infty} |H(2\pi j\nu)|^2 d\nu \tag{4.21}$$

in taking the last steps we have applied Parseval's theorem and changed from a double sided $S_{d_I}(-\infty < \nu < +\infty)$ to a one-sided S_I: twice the integral from $0 < \nu < \infty$ to accomodate physically real frequencies.

Chapter 5

Coherence phenomena

5.1 The Visibility function

The coherence phenomenon is directly coupled to correlation, and the degree of coherence of an EM-wave field $\tilde{E}(\vec{r},t)$ can be quantitativily described by employing the auto- and cross-correlation technique for the analysis of a stochastic process.
The electric vector of the wave field at a position $\vec{r}$ at time t, $\tilde{E}(\vec{r},t)$, is a complex quantity, denoting the amplitude and phase of the field. To assess the coherence phenomenon, the question to be answered is: how do the nature of the source and the geometrical configuration of the situation relate to the resulting phase correlation between two laterally spaced points in the wave field?
This brings to mind Young's interference experiment in which a primary monochromatic source S illuminates two pinholes in an opaque screen, see figure 5.1. The pinholes S_1 and S_2 act as secondary sources, generating a fringe pattern on a distant observation plane Σ_O. If S is an idealized monochromatic point source, the wavelets issuing from any set of apertures S_1 and S_2 will maintain a constant relative phase; they are precisely correlated and therefore mutually fully coherent. On the observation plane Σ_O a well-defined array of stable fringes will result and the radiation field is spatially coherent. At the other extreme, if the pin-

holes S_1 and S_2 are illuminated by separate thermal sources (even with narrow frequency bandwidths), no correlation exists; no fringes will be observable in the observation plane Σ_O and the fields at S_1 and S_2 are said to be incoherent. The generation of interference fringes is seemingly a convenient measure of the degree of coherence of a radiation field. The quality of the fringes produced by an interferometric system can be described quantitativily using the *Visibility function* V:

$$V = \frac{I_{max} - I_{min}}{I_{max} + I_{min}} \tag{5.1}$$

here I_{max} and I_{min} are the irradiances corresponding to the maximum and adjacent minimum in the fringe system.

5.1.1 Young's dual beam interference experiment

To assess the mutual coherence between two positions in a radiation field in a quantitative fashion, consider the situation displayed in figure 5.1, with an extended narrow bandwidth radiation source S, which generates fields $\tilde{E}(\vec{r}_1, t) \equiv \tilde{E}_1(t)$ at S_1 and $\tilde{E}(\vec{r}_2, t) \equiv \tilde{E}_2(t)$ at S_2, respectively. *Interalia: no polarization effects are considered, and therefore a scalar treatment using* $\tilde{E}(\vec{r}, t)$ *will suffice.*

If these two positions in the radiation field are isolated using an opaque screen with two small apertures, we are back to Young's experimental set-up. The two apertures serve as sources of secondary wavelets, which propagate out to some point P on the observation plane Σ_O. The resultant field at P is:

$$\tilde{E}_P(t) \;=\; \tilde{C}_1 \cdot \tilde{E}_1(t - t_1) + \tilde{C}_2 \cdot \tilde{E}_2(t - t_2) \tag{5.2}$$

with $t_1 = r_1/c$ and $t_2 = r_2/c$, r_1 and r_2 representing the pathlengths to P as measured from S_1 and S_2, respectively. This expression shows that the field at the space-time point (P, t) can be determined from the field that existed at S_1 and S_2 at $(t - t_1)$ and $(t - t_2)$, respectively, these being the instants when the light, which is now overlapping at P, first emerged from the apertures. The quantities $\tilde{C}_1$ and $\tilde{C}_2$ are so-called *propagators*,

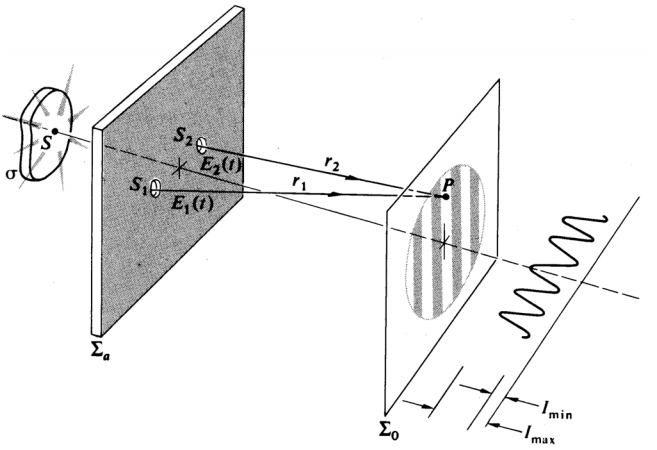

Figure 5.1: *Young's experiment using a quasi-monochromatic source S illuminating two pinholes S_1 and S_2. Credit Hecht(1987).*

they mathematically reflect the alterations in the field resulting from it having transversed either of the apertures. For example, the secondary wavelets issuing from the pinholes in the Young set-up are out of phase by $\pi/2$ radians with the primary wave incident on the aperture screen. In that case, $\tilde{C}_1$ and $\tilde{C}_2$ are purely imaginary numbers equal to $e^{i\pi/2}$.

5.1.2 The mutual coherence function

The resultant irradiance at P, averaged over some time interval which is long compared to the coherence time, is:

$$I = \mathbf{E}\left\{\tilde{E}_P(t) \cdot \tilde{E}_P^*(t)\right\} \tag{5.3}$$

Employing equation (5.2) this can be written as:

$$\begin{aligned} I \;=\; & \tilde{C}_1\tilde{C}_1^*\mathbf{E}\left\{\tilde{E}_1(t-t_1)\cdot\tilde{E}_1^*(t-t_1)\right\} \\ & + \tilde{C}_2\tilde{C}_2^*\mathbf{E}\left\{\tilde{E}_2(t-t_2)\cdot\tilde{E}_2^*(t-t_2)\right\} \\ & + \tilde{C}_1\tilde{C}_2^*\mathbf{E}\left\{\tilde{E}_1(t-t_1)\cdot\tilde{E}_2^*(t-t_2)\right\} \\ & + \tilde{C}_1^*\tilde{C}_2\mathbf{E}\left\{\tilde{E}_1^*(t-t_1)\cdot\tilde{E}_2(t-t_2)\right\} \end{aligned} \tag{5.4}$$

It is now assumed that the wave field is *wide sense stationary* and *ergodic*, see equations (3.2) and (3.3), as is almost universally the case, i.e. the statistical nature of the wave field does not alter with time and the time average is independent of whatever time origin we select. Accordingly, the first two expectation values in equation (5.4) can be rewritten as:

$$I_{S_1} \;=\; \mathbf{E}\left\{\tilde{E}_1(t)\cdot\tilde{E}_1^*(t)\right\} \text{ and } I_{S_2} \;=\; \mathbf{E}\left\{\tilde{E}_2(t)\cdot\tilde{E}_2^*(t)\right\} \tag{5.5}$$

where the time origin was displaced by amounts t_1 and t_2, respectively. The subscripts for the irradiances used here underscore the fact that they refer to the values at points S_1 and S_2. Furthermore, if we introduce the time difference $\tau = t_2 - t_1$, the time origin of the last two terms can be shifted by an amount t_2 yielding:

$$\tilde{C}_1\tilde{C}_2^*\mathbf{E}\left\{\tilde{E}_1(t+\tau)\cdot\tilde{E}_2^*(t)\right\} + \tilde{C}_1^*\tilde{C}_2\mathbf{E}\left\{\tilde{E}_1^*(t+\tau)\cdot\tilde{E}_2(t)\right\} \tag{5.6}$$

This expression comprises a complex quantity plus its own complex conjugate and is therefore equal to twice its *Real* part:

$$2\;\mathit{Re}\left[\tilde{C}_1\tilde{C}_2^*\mathbf{E}\left\{\tilde{E}_1(t+\tau)\cdot\tilde{E}_2^*(t)\right\}\right] \tag{5.7}$$

As noted before, the $\tilde{C}$-coefficients are purely imaginary, i.e. $\tilde{C}_1\tilde{C}_2^* = \tilde{C}_1^*\tilde{C}_2 = |\tilde{C}_1||\tilde{C}_2|$.
The expectation value contained in expression (5.7) is a cross-correlation function, which is denoted by:

$$\tilde{\Gamma}_{12}(\tau) = \mathbf{E}\left\{\tilde{E}_1(t+\tau)\cdot\tilde{E}_2^*(t)\right\} \tag{5.8}$$

This equation is referred to as the **mutual coherence function** of the wave field at positions S_1 and S_2. Making use of the definitions above, equation (5.4) now takes the form:

$$I \;=\; |\tilde{C}_1|^2 I_{S_1} + |\tilde{C}_2|^2 I_{S_2} + 2|\tilde{C}_1||\tilde{C}_2|\; Re\, \tilde{\Gamma}_{12}(\tau) \tag{5.9}$$

The terms $|\tilde{C}_1|^2 I_{S_1}$ and $|\tilde{C}_2|^2 I_{S_2}$ are the irradiance at P, arising when one or the other of the apertures is open alone: either $\tilde{C}_1 = 0$ or $\tilde{C}_2 = 0$. Denoting these irradiances as I_1 and I_2 one can write:

$$I \;=\; I_1 + I_2 + 2|\tilde{C}_1||\tilde{C}_2|\; Re\, \tilde{\Gamma}_{12}(\tau) \tag{5.10}$$

If S_1 and S_2 are made to coincide, the mutual coherence function becomes the autocorrelation function:

$$\tilde{\Gamma}_{11}(\tau) \;=\; \tilde{R}_{E_1}(\tau) \;=\; \mathbf{E}\left\{\tilde{E}_1(t+\tau)\cdot\tilde{E}_1^*(t)\right\} \tag{5.11}$$

or:

$$\tilde{\Gamma}_{22}(\tau) \;=\; \tilde{R}_{E_2}(\tau) \;=\; \mathbf{E}\left\{\tilde{E}_2(t+\tau)\cdot\tilde{E}_2^*(t)\right\} \tag{5.12}$$

One can imagine that two wave trains emerge from these coalesced pinholes and somehow pick up a relative phase delay τ. These autocorrelation functions are also called *self-coherence functions.*
In the situation at hand $\tau = 0$, since the optical path difference (*shorthand:* OPD) goes to zero. Hence:

$$\begin{aligned}\Gamma_{11}(0) \;&=\; \tilde{R}_{E_1}(0) \;=\; \mathbf{E}\left\{\tilde{E}_1(t)\cdot\tilde{E}_1^*(t)\right\} \;=\; \mathbf{E}\left\{|\tilde{E}_1(t)|^2\right\} \;=\; I_{S_1} \text{ and}\\ \Gamma_{22}(0) \;&=\; \tilde{R}_{E_2}(0) \;=\; \mathbf{E}\left\{\tilde{E}_2(t)\cdot\tilde{E}_2^*(t)\right\} \;=\; \mathbf{E}\left\{|\tilde{E}_2(t)|^2\right\} \;=\; I_{S_2}\end{aligned} \tag{5.13}$$

The expressions in (5.13) represent the average intensity (irradiance) of the radiation field at positions S_1 and S_2 respectively, this was already introduced and discussed in sections 2.4 and 2.5.

5.2 The complex degree of coherence

From equation (5.10) and the selfcoherence functions one can now write:

$$|\tilde{C}_1||\tilde{C}_2| \;=\; \frac{(I_1 \cdot I_2)^{\frac{1}{2}}}{[\,\Gamma_{11}(0) \cdot \Gamma_{22}(0)\,]^{\frac{1}{2}}} \tag{5.14}$$

Hence, the normalized expression for the mutual coherence function can now be defined as:

$$\tilde{\gamma}_{12}(\tau) \;\equiv\; \frac{\tilde{\Gamma}_{12}(\tau)}{[\,\Gamma_{11}(0) \cdot \Gamma_{22}(0)\,]^{\frac{1}{2}}} \;=\; \frac{\mathbf{E}\left\{\tilde{E}_1(t+\tau) \cdot \tilde{E}_2^*(t)\right\}}{\left[\,\mathbf{E}\left\{|\tilde{E}_1(t)|^2\right\} \cdot \mathbf{E}\left\{|\tilde{E}_2(t)|^2\right\}\,\right]^{\frac{1}{2}}} \tag{5.15}$$

This quantity is referred to as the **complex degree of coherence**, equation (5.10) can now be recast into its final form:

$$I \;=\; I_1 \;+\; I_2 \;+\; 2(I_1 \cdot I_2)^{\frac{1}{2}}\; Re\;\tilde{\gamma}_{12}(\tau) \tag{5.16}$$

which is the *general interference law for a partially coherent radiation field.*
The quantity $\tilde{\gamma}_{12}(\tau)$ *simultaneously* gives a measure of the *spatial coherence* by comparison of two locations in space (S_1 and S_2 in the above case) and the *coherence in the time domain* by accounting for a time lag τ between both signals.
$\tilde{\gamma}_{12}(\tau)$ is a complex variable and can be written as:

$$\tilde{\gamma}_{12}(\tau) \;=\; |\tilde{\gamma}_{12}(\tau)| \cdot e^{i\psi_{12}(\tau)} \tag{5.17}$$

From equation (5.15) and the Schwarz inequality it is clear that $0 \;\leq\; |\tilde{\gamma}_{12}(\tau)| \;\leq 1$. The phase angle $\psi_{12}(\tau)$ of $\tilde{\gamma}_{12}(\tau)$ relates to the phase angle between the fields at S_1 and S_2 and the phase angle difference concomitant with the OPD in P resulting in the time lag τ, as shown in equation (5.8). For quasi-monochromatic radiation at a mean wavelength $\overline{\lambda}$ and frequency $\overline{\nu}$, the phase difference ϕ due to the OPD can be expressed as:

$$\phi \;=\; \frac{2\pi}{\overline{\lambda}}(r_2 - r_1) \;=\; 2\pi\overline{\nu}\tau \tag{5.18}$$

If we designate a phase angle $\alpha_{12}(\tau)$ between the fields at S_1 and S_2, we have
$\psi_{12}(\tau) = [\alpha_{12}(\tau) - \phi)]$.
Hence:

$$Re\, \tilde{\gamma}_{12}(\tau) \;=\; |\tilde{\gamma}_{12}(\tau)| \cos\left[\alpha_{12}(\tau) - \phi\right] \tag{5.19}$$

Substitution of this expression in the interference law for partially coherent radiation given in equation (5.16) yields for the intensity observed at point P on the observation plane Σ_O:

$$I \;=\; I_1 \;+\; I_2 \;+\; 2(I_1 \cdot I_2)^{\frac{1}{2}}\; |\tilde{\gamma}_{12}(\tau)| \cos\left[\alpha_{12}(\tau) - \phi\right] \tag{5.20}$$

The maximum and minimum values of I occur if the cosine term in equation (5.20) equals $+1$ and -1, respectively. The Visibility V (see definition (5.1)) at position P can therefore be expressed as:

$$V = \frac{2(I_1 \cdot I_2)^{\frac{1}{2}}}{I_1 + I_2} |\tilde{\gamma}_{12}(\tau)| \tag{5.21}$$

In practice, frequently things are (or can be) adjusted in such a way that $I_1 = I_2$, giving rise to the following simplified expressions for the total irradiance I and Visibility V:

$$I \;=\; 2I_0 \left\{1 + |\tilde{\gamma}_{12}(\tau)| \cos\left[\alpha_{12}(\tau) - \phi\right]\right\} \text{ and } V \;=\; |\tilde{\gamma}_{12}(\tau)| \tag{5.22}$$

Note that in this case *the modulus of the complex degree of coherence is identical to the visibility of the fringes* ! This then provides an experimental means of obtaining $|\tilde{\gamma}_{12}(\tau)|$ from the resultant fringe pattern. Moreover, the off-axis shift in the location of the central fringe (no OPD $\rightarrow \phi = 0$) is a measure of $\alpha_{12}(\tau)$, the relative retardation in phase of the fields at S_1 and S_2. Thus, measurements of the visibility and the fringe position yield both the amplitude and phase of the complex degree of coherence.

5.3 Temporal coherence

Temporal coherence is characterised by the coherence time τ_c . The value of τ_c follows from the finite bandwidth of the radiation source un-

der consideration as indicated in section 2.1.

These effects can quantitatively be assessed with the aid of the Wiener-Khinchine theorem:

$$S(\nu) = \int_{-\infty}^{+\infty} R(\tau) \cdot e^{-2\pi i \nu \tau} d\tau \tag{5.23}$$

$$R(\tau) = \int_{-\infty}^{+\infty} S(\nu) \cdot e^{2\pi i \nu \tau} d\nu \tag{5.24}$$

Take as an example a Gaussian shaped spectral line profile, i.e.

$$S(\nu) \sim e^{-\left(\frac{\nu}{\Delta\nu}\right)^2} \Leftrightarrow R(\tau) \sim e^{-\left(\frac{\tau}{\tau_c}\right)^2} \tag{5.25}$$

As can be seen from the FT (indicated by $\Leftrightarrow$ in expression(5.25), the wave packet corresponding to this line profile has an autocorrelation function that is also Gaussian with a characteristic width τ_c, moreover the *autocorrelation* $R(\tau)$ equals the *autocovariance* $C(\tau)$. This corresponds to a wave train with a Gaussian shaped envelope for the wave amplitude, see again figure (3.1) in section 2.1.
For spectroscopic measurements at infrared and shorter wavelengths one can directly disperse the incoming radiation beam with the aid of a wavelength dispersive device, like for instance a transmission or a reflection grating, and measure the resulting intensity distribution (i.e. the spectrum). However for spectroscopy at radio and submillimeter wavelengths one employs an indirect method. The incoming wave signal is fed into a *correlator* that produces the temporal coherence function $R(\tau)$, a subsequent FT of this function yields the spectral distribution $S(\nu)$ by virtue of the Wiener-Khinchine relation.

5.4 Longitudinal coherence

Associated with the coherence time τ_c is the so-called coherence length $l_c = c\tau_c$.

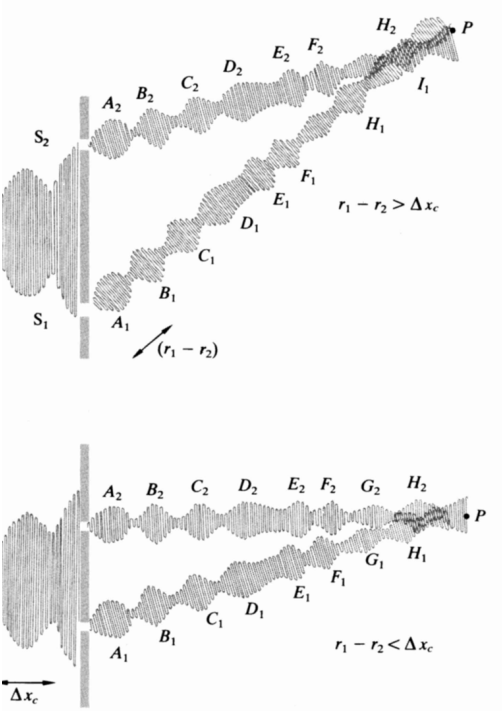

Figure 5.2: *The influence of the coherence length on the interference pattern of two diffracted coherent thermal radiation sources S_1 and S_2. Credit Hecht(1987).*

Problem: Show that the coherence length can also be expressed as $l_c = \lambda^2/\triangle\lambda$ in which $\triangle\lambda$ refers to the equivalent of $\triangle\nu$ in the wavelength domain.

Now consider an EM-wave that propagates along a vector $\vec{r}$, and mark two positions P_1 and P_2 on this line of propagation at a mutual distance R_{12}. If $R_{12} \ll l_c$, there will be a strong correlation between the

EM-fields at P_1 and P_2 and as a consequence interference effects will be possible. In the case of $R_{12} \gg l_c$, no interference effects are possible. This effect (i.e. potential interference *yes* or *no*) relates to the so-called *longitudinal correlation* or *longitudinal spatial coherence*.
This effect can be clearly demonstrated by considering the wave trains in Young's interference experiment (see figure 5.2). The diffracted beams emanating from S_1 and S_2, which are coherent radiation sources, cause an interference pattern. However, in the case of large path differences the interference contrast will diminish, since corresponding wave packets in the stochastic signal no longer overlap (see figure 5.2: packet H_1 and $H_2 \Rightarrow$ packet I_1 and H_2).

5.5 Spatial or lateral coherence

Spatial coherence (also: lateral coherence or lateral correlation) has to do with the spatial extent of the radiation source.

Problem: Prove that for $\tau \ll \tau_c$:

$$\tilde{\gamma}_{12}(\tau) = \tilde{\gamma}_{12}(0) \cdot e^{2\pi i \bar{\nu} \tau} \tag{5.26}$$

with $|\tilde{\gamma}_{12}(\tau)| = |\tilde{\gamma}_{12}(0)|$ and a *fixed phase difference* $\alpha_{12}(\tau) = 2\pi\bar{\nu}\,\tau$, $\bar{\nu}$ represents the average frequency of the wave carrier.

In the following treatment of spatial coherence, it is implicitly assumed that the frequency bandwidth of the radiation source is suffiently narrow that the comparison between two points with respect to spatial coherence occurs at times differing by $\triangle t \ll \tau_c$
Query What is the quantitative relation between the *brightness distribution* of the spatially extended radiation source and the resulting *phase correlation* between two positions in the radiation field?

Approach Consider again Young's experiment for the case that the radiation source S is extended and illuminates the pinholes S_1 and S_2 (actually shown in figure 5.1). In the observers plane Σ, the interference is given by the expectation value of the product $\tilde{E}_1(t) \cdot \tilde{E}_2^*(t) = \mathbf{E}\{\tilde{E}_1(t) \cdot \tilde{E}_2^*(t)\} = \tilde{\Gamma}_{12}(0)$ with the subscripts 1 and 2 referring to the positions P_1

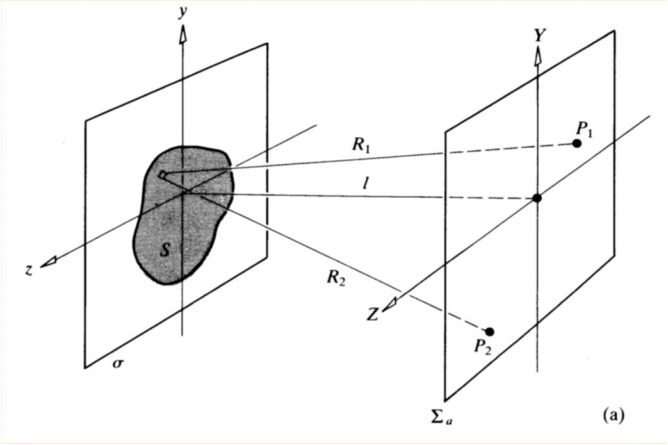

Figure 5.3: *Relating $\tilde{\gamma}_{12}(0)$ to the brightness distribution of an extended radiation source S: configuration for demonstrating the Van Cittert-Zernike theorem. Credit Hecht (1987).*

and P_2 in the Σ-plane. If $\tilde{E}_1$ and $\tilde{E}_2$ are uncorrelated, then $|\tilde{\Gamma}_{12}(0)| = 0$. In the case of full correlation $|\tilde{\gamma}_{12}| \left(= |\tilde{\Gamma}_{12}(0)|/(I_1 \cdot I_2)^{\frac{1}{2}}\right) = 1$, for partial correlation one has $0 < |\tilde{\gamma}_{12}(0)| < 1$.
The extended source in Young's experiment is a collection of non-coherent infinitesimal radiators, this obviously reduces the contrast in the interferogram. This contrast can be observed and is described by the afore mentioned *Visibility function V*:

$$V = \frac{I_{max} - I_{min}}{I_{max} + I_{min}} = |\tilde{\gamma}_{12}(0)| \tag{5.27}$$

5.6 The Van Cittert-Zernike theorem

So how does one relate $\tilde{\gamma}_{12}(0)$ (or $\tilde{\Gamma}_{12}(0)$) to the brightness distribution of the extended radiation source S?
This can be done in the following way (see figure 5.3). Locate S, a QM-

incoherent source, in a plane σ, with an intensity distribution $I(y,z)$. Consider next the observation plane Σ parallel to σ, l is perpendicular to both planes (coincident with the X-axis) connecting the centre of the extended source ($y=0, z=0$) to the zero reference in Σ ($Y=0, Z=0$). Select two positions P_1 and P_2. The objective is to describe the value of $\tilde{\gamma}_{12}(0)$ in this plane, i.e. the coherence of the radiation field in Σ. To assess this, consider a small (infinitesimal) radiation element dS in the source at distances R_1 and R_2 from P_1 and P_2 respectivily. Suppose now that S is *not a source* but *an aperture* of identical size and shape, and suppose that $I(y,z)$ is not a description of the irradiance (or intensity distribution) but, instead, its functional form corresponds to the *field distribution* across that aperture. In other words imagine that there is a transparancy at the aperture with amplitude transmission characteristics that correspond functionally to the irradiance distribution $I(y,z)$. Now imagine that this aperture is illuminated by a spherical wave converging towards the fixed point P_2, so that a diffraction pattern will result centered at P_2. This diffracted field distribution, normalised to unity at P_2, is everywhere (e.g. at P_1) equal to the value of $\tilde{\gamma}_{12}(0)$ at that point. This is the *Van Cittert-Zernike theorem.*
In the limit that R_1 and R_2 are much larger than the source diameter and the relevant part of the Σ-plane we have the equivalent of Fraunhofer diffraction, this condition is practically always satisfied for astronomical observations. In that case, the van Cittert-Zernike theorem can be expressed mathematically as:

$$\tilde{\Gamma}(\vec{r}) = \int\int_{\text{source}} I(\vec{\Omega}) \cdot e^{\frac{2\pi i \vec{\Omega}.\vec{r}}{\lambda}} d\vec{\Omega} \tag{5.28}$$

$$I(\vec{\Omega}) = \lambda^{-2} \int\int_{\Sigma\text{-plane}} \tilde{\Gamma}(\vec{r}) \cdot e^{-\frac{2\pi i \vec{\Omega}.\vec{r}}{\lambda}} d\vec{r} \tag{5.29}$$

$I(\vec{\Omega})$ is the intensity distribution of the extended radiation source as a function of a unit direction vector $\vec{\Omega}$ as seen from the observation plane Σ. Taking the centre of the extended radiation source S as the zero-reference for $\vec{\Omega}$ (coincident with the central axis l in figure (5.3)) and assuming a relativily small angular extent of the source one may write $I(\vec{\Omega}) = I(\theta_y, \theta_z)$ and $d\vec{\Omega} = d\theta_y d\theta_z$, where θ_y and θ_z represent two or-

thogonal angular coordinate axes across the source starting from the zero-reference of $\vec{\Omega}$.
$\tilde{\Gamma}(\vec{r})$ is the coherence function in the Σ-plane, the vector $\vec{r}$ represents an arbitrary baseline $\vec{r}(X,Y)$ in this plane with $d\vec{r} = dYdZ$ (in the above example $\vec{P_1P_2} = \vec{r}_{P_1} - \vec{r}_{P_2}$).
Expressions (5.28) and (5.29) for $\tilde{\Gamma}(\vec{r})$ and $I(\vec{\Omega})$ show that they are linked through a Fourier transform, except for the scaling with the wavelength λ. This scaling might be perceived as a "true" Fourier transform with the *conjugate variables* $\vec{\Omega}$ and $\vec{r}/\lambda$, i.e. by expressing $\vec{r}$ in units of the wavelength λ, *writing the van Cittert-Zernike theorem as the Fourier pair*:

$$I(\vec{\Omega}) \quad \Leftrightarrow \quad \tilde{\Gamma}(\vec{r}/\lambda) \tag{5.30}$$

The complex *spatial* degree of coherence, $\tilde{\gamma}(\vec{r})$, follows from:

$$\tilde{\gamma}(\vec{r}) = \frac{\tilde{\Gamma}(\vec{r})}{\int\int_{\text{source}} I(\vec{\Omega})d\vec{\Omega}} \tag{5.31}$$

i.e. by normalising on the total source intensity.
Note: Although the extended source S is spatially incoherent, there still exists a partially correlated radiation field at e.g. positions P_1 and P_2, since all individual source elements contribute to a specific location P in the Σ-plane.
End of Note

For a derivation of the Van Cittert-Zernike relations, consider the geometry given in figure 5.4.
The observation plane Σ contains the baseline vector $\vec{r}(Y,Z)$ and is perpendicular to the vector pointing at the centre of the radiation source. The angular coordinates θ_y and θ_z across the source (see above) correspond to the linear coordinates of the unit direction vector $\vec{\Omega}(\Omega_X, \Omega_Y, \Omega_Z)$, i.e. the direction cosines of $\vec{\Omega}$ relative to the X,Y,Z coordinate system $(\Omega_X^2 + \Omega_Y^2 + \Omega_Z^2 = 1)$. The spatial coherence of the EM-field between the two positions 1 (for convenience chosen in the origin) and 2 is the outcome of a correlator device that produces the output $\mathbf{E}\left\{\tilde{E}_1(t) \cdot \tilde{E}_2^*(t)\right\}$.
In reality positions 1 and 2 are not point like, they represent *radio an-*

tennae or *optical reflectors*, this issue will be addressed later on. From the geometry displayed in figure 5.4, regarding the Van Cittert-Zernike relations, one can note the following:
$\Rightarrow$ If $I(\vec{\Omega}) = I_0\delta(\vec{\Omega})$, i.e. a point source on the X-axis, the Van Cittert-Zernike relation yields $|\tilde{\Gamma}(\vec{r})| = I_0$ and $|\tilde{\gamma}(\vec{r})| = 1$: a plane wave hits the full YZ-plane everywhere at the same time, full coherence is preserved (by definition) on a plane wave front.
$\Rightarrow$ Next, let's consider an infinitesimal source element in the direction $\vec{\Omega}_0 \Longrightarrow I_0\delta(\vec{\Omega}-\vec{\Omega}_0)$. The projection of $\vec{\Omega}_0$ on the $\Sigma-$plane is $\vec{\Omega}'_0(\Omega_Y, \Omega_Z)$. There will now be a difference in path length between positions 1 and 2 given by the projection of $\vec{r}$ on $\vec{\Omega}_0$, i.e. $\vec{r}.\vec{\Omega}'_0 = \Omega_Y Y + \Omega_Z Z$. Then:

$$\tilde{E}_1(t) = \tilde{E}_0(t) \cdot e^{2\pi i\nu_0\left(t+\frac{\vec{\Omega}_0\cdot\vec{r}}{c}\right)} = \tilde{E}_0(t) \cdot e^{\left(2\pi i\nu_0 t+\frac{2\pi i\vec{\Omega}_0\cdot\vec{r}}{\lambda}\right)} \tag{5.32}$$

$$\tilde{E}_2^*(t) = \tilde{E}_0(t) \cdot e^{-2\pi i\nu_0 t} \tag{5.33}$$

Therefore:

$$\mathbf{E}\left\{\tilde{E}_1(t)\cdot\tilde{E}_2^*(t)\right\} = \mathbf{E}\left\{|\tilde{E}_0(t)|^2\right\}\cdot e^{\frac{2\pi i\vec{\Omega}_0\cdot\vec{r}}{\lambda}} = I_0(\vec{\Omega}_0)\cdot e^{\frac{2\pi i\vec{\Omega}_0\cdot\vec{r}}{\lambda}} \tag{5.34}$$

Integration over the full source extent(straight forward integration, since all source elements are spatially uncorrelated) yields:

$$\tilde{\Gamma}(\vec{r}) = \int\int_{\text{source}} I_0(\vec{\Omega})\cdot e^{2\pi i\vec{\Omega}.\vec{r}/\lambda} d\vec{\Omega} \tag{5.35}$$

$$\tilde{\gamma}(\vec{r}) = \frac{\int\int_{\text{source}} I_0(\vec{\Omega})\cdot e^{2\pi i\vec{\Omega}.\vec{r}/\lambda} d\vec{\Omega}}{\int\int_{\text{source}} I_0(\vec{\Omega}) d\vec{\Omega}} \tag{5.36}$$

The meaning of this relationship is, in physical terms, that $\tilde{\Gamma}(\vec{r})$ at a certain point represents a *single* Fourier component (with baseline $\vec{r}$) of the intensity distribution of the source with strength $\tilde{\Gamma}(\vec{r})d\vec{r}$. A short baseline (small $|\vec{r}|$) corresponds to a low frequency (*spatial frequency!*) component in the brightness distribution $I(\theta_y, \theta_z)$, i.e. *coarse* structure, large values of $|\vec{r}|$ correspond to *fine* structure in $I(\theta_y, \theta_z)$. The *diffraction limited* resolution in *aperture synthesis* corresponds to:

$$|\vec{r}_{max}| = L_{max} \Longrightarrow \theta_{min} = \frac{\lambda}{2L_{max}} \tag{5.37}$$

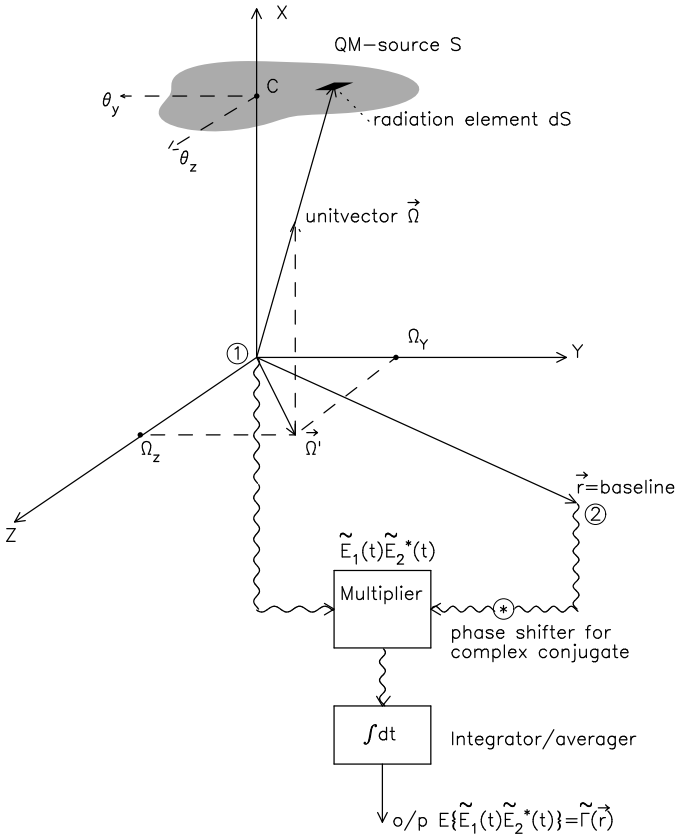

Figure 5.4: *Van Cittert Zernike relation: reference geometry.*

The factor 2 in the denominator of the expression for θ_{min} follows from the rotation symmetry in aperture synthesis.

Example Consider a one dimensional case. This can be done by taking the slit source of uniform intensity shown in figure 5.5 , slit width b and running coordinate ξ. The observation plane Σ, running coordinate y,

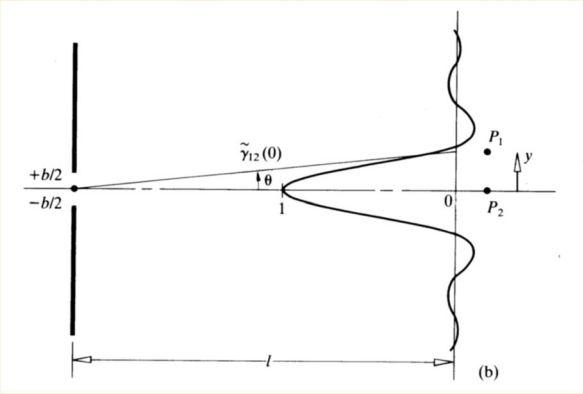

Figure 5.5: *The coherence function $\tilde{\gamma}_{12}(0)$ for a uniform slit source. Credit Hecht (1987).*

is located at large distance l from the slit source (i.e. the Fraunhofer limit is applicable). The source function can be expressed as the window function $\Pi\left(\xi/b\right)$, in angular equivalent $\Pi\left(\beta/\beta_0\right)$, with $\beta_0 = b/l$. Application of the Van Cittert-Zernike theorem $I(\vec{\Omega}) \Leftrightarrow \tilde{\Gamma}(\vec{r}/\lambda)$ yields:

$$\Pi\left(\frac{\beta}{\beta_0}\right) \Leftrightarrow \beta_0 \mathrm{sinc}\left(\frac{y\beta_0}{\lambda}\right) = \beta_0 \mathrm{sinc}\left(\frac{yb}{\lambda l}\right) \tag{5.38}$$

with $\mathrm{sinc}(x) = \sin \pi x/(\pi x)$. The modulus of the normalised complex coherence function becomes:

$$|\tilde{\gamma}(y)| = \left|\frac{\beta_0 \mathrm{sinc}\left[(yb)/(\lambda l)\right]}{\beta_0}\right| = \left|\mathrm{sinc}\left(\frac{yb}{\lambda l}\right)\right| = V \Rightarrow \text{Visibility} \tag{5.39}$$

Note that:

- *Enlarging* b with a factor 2, *shrinks* the coherence function with the same factor. This is of course a direct consequence of the scaling law under Fourier transform.

- The width of the coherence function follows from: $y\beta_0/\lambda \approx 1 \Rightarrow y = \lambda/\beta_0$. If the radiation source exhibits a smooth brightness distribution over the angle $\beta_0 = \Delta$ radians, as is the case with the slit source, then $\gamma(y)$ also displays a smooth distribution over a distance of λ/Δ meters.

- If the brightness structure of a radiation source covers a wide range of angular scales, say from a largest angular scale Δ to a smallest angular scale δ (in radians), then the spatial coherence function shows a finest detail of λ/Δ and a maximum extent of $\approx \lambda/\delta$ in meters.

5.7 Etendue of coherence

Consider the two-dimensional case of a circular source of uniform intensity with an angular diameter θ_s, the source brightness distribution can then be described as a circular two-dimensional window function: $I(\vec{\Omega}) = \Pi(\theta/\theta_s)$. To compute the complex degree of coherence in the observation plane Σ take again two positions, position 1 in the centre (origin, as before) and position 2 at a distance ρ from this centre point. Applying the van Cittert-Zernike theorem yields for $\tilde{\Gamma}(\rho)$:

$$\Pi\left(\frac{\theta}{\theta_s}\right) \Leftrightarrow \tilde{\Gamma}(\rho/\lambda) = \frac{(\theta_s/2)J_1(\pi\theta_s\rho/\lambda)}{\rho/\lambda} \tag{5.40}$$

where J_1 represents the Bessel function of the first kind. Normalisation to the source brightness, through division by $(\pi\theta_s^2)/4$, yields the expression for the complex degree of coherence:

$$\tilde{\gamma}(\rho) = \frac{2J_1(\pi\theta_s\rho/\lambda)}{\pi\theta_s\rho/\lambda} \tag{5.41}$$

The modulus of the complex degree of coherence is therefore:

$$|\tilde{\gamma}(\rho)| = \left|\frac{2J_1(u)}{u}\right| \tag{5.42}$$

with the argument of the Bessel function $u = \pi\theta_s\rho/\lambda$. One can derive the extent of the coherence in the observation plane Σ by evaluating $J_1(u)$. By selecting $u = 2$, $|\tilde{\gamma}(\rho)| = J_1(2) = 0.577$, i.e. the coherence in Σ remains significant for $u \leq 2$, or $\rho \leq 2\lambda/(\pi\theta_s)$. The area S in Σ over which the coherence remains significant equals $\pi\rho^2 = 4\lambda^2/(\pi\theta_s^2)$. In this expression, $\pi\theta_s^2/4$ equals the solid angle Ω_{source} subtended by the radiation source. Significant coherence will thus exist if the following condition is satisfied:

$$\epsilon = S\Omega_{\text{source}} \leq \lambda^2 \tag{5.43}$$

The condition $\epsilon = S\Omega_{\text{source}} = \lambda^2$ is called the *Etendue of Coherence*, to be fulfilled if coherent detection is required!

Example Consider a red giant star, of radius $r_0 = 1.5$ x 10^{11} meter, at a distance of 10 parsec. For this object $\theta_s = 10^{-6}$ radians. If this object is observed at $\lambda = 0.5\mu$m, the value of the coherence radius ρ, on earth, on a screen normal to the incident beam is $\rho = 2\lambda/(\pi\theta_s) = 32$ cm. In the infrared, at $\lambda = 25\mu$m, the radius ρ is increased fifty fold to ≈ 15 meter. In the radio domain, say at $\lambda = 6$ cm, $\rho \approx 35$ km.

In general, *good coherence* means a Visibility of 0.88 or better. For a uniform circular source this occurs for $u = 1$, that is when $\rho = 0.32\lambda/\theta$. A narrow-bandwidth uniform radiation source at a distance R away yields:

$$\rho = 0.32(\lambda R)/D \tag{5.44}$$

This expression is very convenient to quickly estimate the required physical parameters in an interference or diffraction experiment. For example, if one puts a red filter over a 1-mm-diameter disk-shaped flashlight source and stands back 20 meters from it, then $\rho = 3.8$ mm, where the mean wavelength is taken at 600 nm. This means that a set of apertures spaced at about 4 mm or less should produce clear fringes. Evidently the area of coherence increases with the distance R, this is why one can always find a distant bright street light to use as a convenient source.
Important: Remember, as stated in section 1.3.1, that throughout the treatment of spatial coherence it was assumed that the comparison between the two points occurs at times differing by a $\Delta t \ll \tau_c$. If this

condition is not fulfilled, for example because the frequency bandwidth of the radiation source is too large, interferometric measurements will not be possible (see section on temporal coherence). Frequency filtering will then be required to reduce the bandwidth of the source signal, i.e. make it more monochromatic.

5.8 Aperture synthesis

As already indicated above, the positions 1 and 2 in the observation plane Σ are in practise not pointlike, but encompass a finite aperture in the form of a telescope element of finite size, say a radio dish with diameter D. This dish has then a diffraction sized beam of λ/D. In that case the Van Cittert-Zernike relation needs to be "weighted" with the telescope element (single dish) transfer function $H(\vec{\Omega})$. For a circular dish antenna $H(\vec{\Omega})$ is the *Airy brightness function*, well known from the diffraction of a circular aperture and detailed in a previous section. The Van Cittert-Zernike relations now become:

$$\tilde{\Gamma}'(\vec{r}) = \int\int_{\text{source}} I(\vec{\Omega}) \cdot H(\vec{\Omega}) \cdot e^{\frac{2\pi i \vec{\Omega}\cdot\vec{r}}{\lambda}} d\vec{\Omega} \tag{5.45}$$

$$I(\vec{\Omega}) \cdot H(\vec{\Omega}) = \lambda^{-2} \int\int_{\Sigma\text{-plane}} \tilde{\Gamma}'(\vec{r}) \cdot e^{-\frac{2\pi i \vec{\Omega}\cdot\vec{r}}{\lambda}} d\vec{r} \tag{5.46}$$

The field of view scales with λ/D, e.g. if λ decreases the synthesis resolution improves but the field of view reduces proportionally!

So, in aperture synthesis the incoming beams from antenna dish 1 and antenna dish 2 are fed into a *correlator (multiplier)* that produces as output the product $\tilde{E}_1(t) \cdot \tilde{E}_2^*(t)$. This output is subsequently fed into an *integrator/averager* that produces the expectation value $\mathbf{E}\left\{\tilde{E}_1(t) \cdot \tilde{E}_2^*(t)\right\} = \tilde{\Gamma}'(\vec{r})$. By applying the Fourier transform given in (5.46), and correcting for the beam profile of the single dish $H(\vec{\Omega})$, the source brightness distribution $I(\vec{\Omega})$ can be reconstructed.

Important: Indirect imaging with an aperture synthesis system is limited to measuring image details within the *single pixel* defined by the beam profile of an individual telescope element, i.e. a single dish!

Chapter 6

Correlating the instantaneous intensity

6.1 The autocorrelation of $I(t)$

The autocorrelation of the instantaneous intensity is given by:

$$\begin{aligned} R_I(\tau) &= \mathbf{E}\{I(t+\tau)\cdot I(t)\} = \overline{I(t)\cdot I(t+\tau)} = \\ &= \overline{\tilde{E}(t)\cdot\tilde{E}^*(t)\cdot\tilde{E}(t+\tau)\cdot\tilde{E}^*(t+\tau)} \end{aligned} \tag{6.1}$$

To evaluate expression (6.1), expand this in the real and imaginary parts of the analytic signal $\tilde{E}(t)$:

$$\begin{aligned} &\overline{\tilde{E}(t)\cdot\tilde{E}^*(t)\cdot\tilde{E}(t+\tau)\cdot\tilde{E}^*(t+\tau)} = \\ &= \overline{Re^2\tilde{E}(t)\cdot Re^2\tilde{E}(t+\tau)} + \overline{Im^2\tilde{E}(t)\cdot Im^2\tilde{E}(t+\tau)} + \\ &+ \overline{Re^2\tilde{E}(t)\cdot Im^2\tilde{E}(t+\tau)} + \overline{Im^2\tilde{E}(t)\cdot Re^2\tilde{E}(t+\tau)} \end{aligned} \tag{6.2}$$

These four separate real correlation functions can be evaluated separately by using the following relation for the Gaussian distributed random vari-

able $a(t)$:

$$\overline{a^2(t)\cdot a^2(t+\tau)} = \overline{a^2(t)}\cdot\overline{a^2(t+\tau)} + 2\left(\overline{a(t)\cdot a(t+\tau)}\right)^2 \tag{6.3}$$

Using the equalities (prove this):

$$\begin{aligned}
\overline{Re^2\tilde{E}(t)} \;=\; \overline{Im^2\tilde{E}(t)} \;=\; \overline{Re^2\tilde{E}(t+\tau)} \;&=\; \overline{Im^2\tilde{E}(t+\tau)} \;=\; \frac{1}{2}\bar{I} \\
\overline{Re\tilde{E}(t)\cdot Re\tilde{E}(t+\tau)} \;=\; \overline{Im\tilde{E}(t)\cdot Im\tilde{E}(t+\tau)} \;&=\; \frac{1}{2}\,Re[\tilde{\Gamma}_{11}(\tau)] \\
-\overline{Re\tilde{E}(t)\cdot Im\tilde{E}(t+\tau)} \;=\; \overline{Im\tilde{E}(t)\cdot Re\tilde{E}(t+\tau)} \;&=\; \frac{1}{2}\,Im[\tilde{\Gamma}_{11}(\tau)]
\end{aligned} \tag{6.4}$$

in which $\tilde{\Gamma}_{11}(\tau)$ is the complex autocorrelation (*selfcoherence*) function of the analytic signal $\tilde{E}(t)$, i.e. $\tilde{\Gamma}_{11}(\tau) = \overline{\tilde{E}(t+\tau)\cdot\tilde{E}^*(t)}$. Thus we get:

$$R_I(\tau) \;=\; \bar{I}^2 + |\tilde{\Gamma}_{11}(\tau)|^2 \equiv \bar{I}^2\left(1 + |\tilde{\gamma}_{11}(\tau)|^2\right) \tag{6.5}$$

with $\tilde{\gamma}_{11}(\tau)$ the normalized autocorrelation coefficient.

This result can also be expressed in a form showing the correlation between the instantaneous fluctuation $\Delta I(t)$ about the mean value $\bar{I}$, i.e. $\Delta I(t) = I(t) - \bar{I}$, this results in:

$$\overline{\Delta I(t)\cdot\Delta I(t+\tau)} \;=\; \bar{I}^2|\tilde{\gamma}_{11}(\tau)|^2 \tag{6.6}$$

In particular for $\tau = 0$ (full correlation $|\tilde{\gamma}_{11}(\tau)| = 1$), one finds again the result for the variance of the intensity, $\overline{(\Delta I)^2} = \bar{I}^2$, which was obtained earlier directly from the intensity probability density function $P(I)$ derived in the previous section. So the intensity fluctuations are also expected to be partially coherent, since the amplitude and phase fluctuations in a thermal signal tend to track each other. An impression of such a fluctuating wave signal is given in figure 6.1, both in absolute value and centered around the avarage value $\overline{I}$: $\triangle I = I - \overline{I}$.

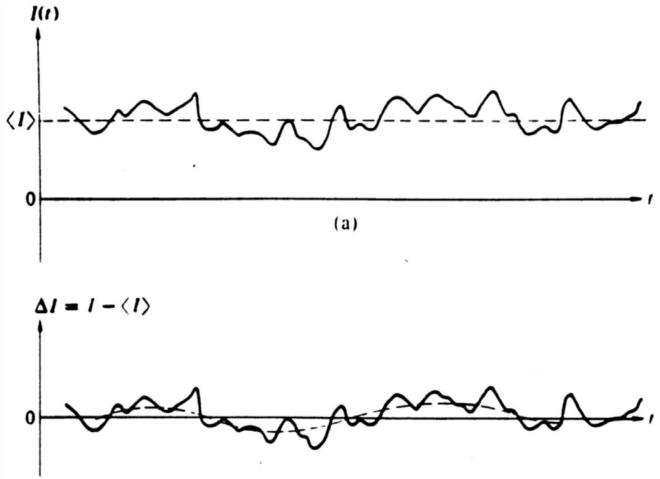

Figure 6.1: *Partially coherent intensity fluctuations in a thermal radiation source. Both absolute values* $\overline{I}$ *and relative values* $\triangle I = I - \overline{I}$ *are shown. Credit Hecht (1987).*

6.2 The influence of *band-limited* radiation sensing

The autocorrelation function derived above refers to the instantaneous intensities. If an optical analytic signal $\tilde{E}(t)$ is considered, the high frequency carrier $\bar{\nu}$ is of the order 10^{15} Hz, the intensity fluctuation varies much slower than $\tilde{E}(t)$, but the time scale is still very short considering $\tau_c \simeq (\Delta\nu)^{-1}$. For atomic transitions $\tau_c = 10^{-9} - 10^{-10}$ s, and thus $\Delta\nu$ is of the order 10^9 Hz. Although τ_c contains of the order of a million oscillations, most measuring devices, like photo-electric devices and the associated electronic correlators, have response times slower than 1 GHz. In practice, therefore, one has to deal with a minimum resolving time T of the band limited sensing devices. If the detection efficiency, i.e. the number of photo-electrons per incident photon is α, with $\alpha < 1$, we can

write a short-term intensity average as:

$$I_T(t) = \frac{\alpha}{T} \int_{-\frac{1}{2}T}^{\frac{1}{2}T} I(t+t')dt' \tag{6.7}$$

It is assumed here that α remains constant over the spectral range $\Delta\nu$ and, more importantly, that any variations in time lag of the photoelectric emission process are negligible. Therefore one also has $\bar{I}_T = \alpha\bar{I}$. The variance $\overline{\Delta I_T^2}$ of this time-averaged signal can be expressed as:

$$[C_I(0)]_T = \overline{(\Delta I_T)^2} = \left(\frac{\alpha\bar{I}}{T}\right)^2 \int\int_{-\frac{1}{2}T}^{\frac{1}{2}T} |\gamma_{11}(t''-t')|^2 \, dt'dt'' \tag{6.8}$$

where use was made of the equality:

$$\overline{\Delta I(t+t') \cdot \Delta I(t+t'')} = \bar{I}^2 |\gamma_{11}(t''-t')|^2 \tag{6.9}$$

The double integral for averaging over the resolving time T can be reduced to a single integral by considering the appropriate surface element in the t'-t'' plane, as shown in figure 6.2. Substituting the surface element $(T-|\tau|)d\tau$ yields:

$$[C_I(0)]_T = \left(\frac{\alpha\bar{I}}{T}\right)^2 \int_{-T}^{T} (T-|\tau|) \cdot |\gamma_{11}(\tau)|^2 d\tau = \alpha^2\bar{I}^2 \left(\frac{\xi(T)}{T}\right) \tag{6.10}$$

with $\xi(T)$ defined as:

$$\xi(T) = \int_{-T}^{T} \left(1 - \frac{|\tau|}{T}\right) \cdot |\gamma_{11}(\tau)|^2 d\tau \tag{6.11}$$

Since $|\gamma_{11}(\tau)| \leq 1$ it follows that $\xi(T) \leq T$. If $T \ll \tau_c (= 1/\Delta\nu)$ one obtains $\xi(T) \simeq T$, i.e. the variance becomes $[C_I(0)]_T = \alpha^2\bar{I}^2$. This result is the same as without averaging and is to be expected, since for T smaller than the coherence time $I_T(t)$ faithfully tracks the instantaneous intensity $I(t)$. If on the other hand $T \gg \tau_c$, which is normally the case,

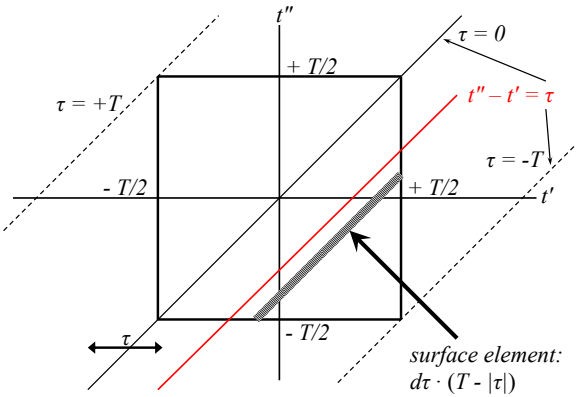

Figure 6.2: *Illustration of the coordinate transformation used to transform the double integral of equation (6.8) to a single integral.*

the integral tends to an upper limit $\xi(\infty) = \int_{-\infty}^{\infty} |\gamma_{11}(\tau)|^2 d\tau \simeq \tau_c$, since $|\gamma_{11}(\tau)|$ vanishes for $|\tau| \gg \tau_c$. In this case the variance becomes:

$$[C_I(0)]_T = \frac{\alpha^2 \bar{I}^2 \tau_c}{T} \tag{6.12}$$

i.e. the original value gets reduced by τ_c/T, hence for sensitive correlation measurements one has the keep the resolving time T as short as possible!

The intensity fluctuation described above arises solely from the analysis of the wave character of the radiation and results from the coherence properties of quasi-monochromatic waves. On top of this fluctuation one has to consider the shot-noise character of the photo-electric emission process, which arises purely from the quantum character of the radiation. The total variance is therefore obtained by adding the shotnoise variance to $[C_I(0)]_T$ above, with the result:

$$\overline{(\Delta I_T)^2} = \alpha \bar{I} \left(1 + \frac{\alpha \bar{I} \xi(T)}{T}\right) \tag{6.13}$$

It makes no difference whether I is treated as the square of the electric

field vector, or as a photon arrival probability density. In a similar way the fluctuation in photon number in a measurement time interval T can be derived to be:

$$\overline{(\Delta n_T)^2} = \alpha \bar{I} T + \alpha^2 \bar{I}^2 T \xi(T) = \bar{n}_T \left(1 + \frac{\bar{n}_T \xi(T)}{T} \right) \tag{6.14}$$

Hence, for $T \ll \tau_c$ one arrives at the characteristic variance for the Bose-Einstein statistics $\overline{(\Delta n_T)^2} = \bar{n}(1 + \bar{n})$. However for $T \gg \tau_c$ the variance changes to $\overline{(\Delta n_T)^2} = \bar{n}(1 + \bar{n}\tau_c/T) = \bar{n}(1 + \alpha \bar{I} \tau_c)$ This is characteristic for n bosons distributed among T/τ_c cells of phase space. This expression demonstrates that the deviation from the Poisson limit is given by the product of the count rate $\alpha \bar{I}$ and the coherence time $\tau_c = \int_{-\infty}^{\infty} |\gamma_{11}(\tau)|^2 d\tau$ determined by the light spectrum.

Physically, the deviation from Poissonian statistics can be attributed to the probability that two wave packets overlap. When this occurs this can lead to four photons (constructive interference) or no photons (destructive interference) and anything in between. It is proper to speak of interference in this situation since these photons are in the same quantum state, and the abnormal density fluctuation occurs in any ensemble of bosons occupying the same quantum state.

The arrival density fluctuation $\overline{(\Delta n)^2}$ can also be derived directly from the probability distribution of the photo-electric counts, which immediately results in the full expression for $\overline{(\Delta n_T)^2}$ as given in equation (6.14). This stipulates again the power of quantum theory relative to the classical wave picture!

The coherence length of the light is the quantity in the wave picture that is complementary to the length of the phase cell in the particle picture. The term $\bar{n}\tau_c/T$ gives the mean number of counts per unit cell of phase space and is the analogue of the degeneration parameter δ encountered in quantum statistics for the number of particles in the same quantum state. The expression $\delta = \bar{n}\tau_c/T = \alpha \bar{I} \tau_c$ derived here is generally applicable, also for 'diluted' photon beams and immediately gives the magnitude of coherence effects in fluctuation analyses. It is of course also evident that in order to have a measurable effect for $\tau_c \simeq 10^{-9}$ s a very large intensity $\bar{I}$ is required for the radiation source. This issue will be elaborated in some detail further on for the case of the

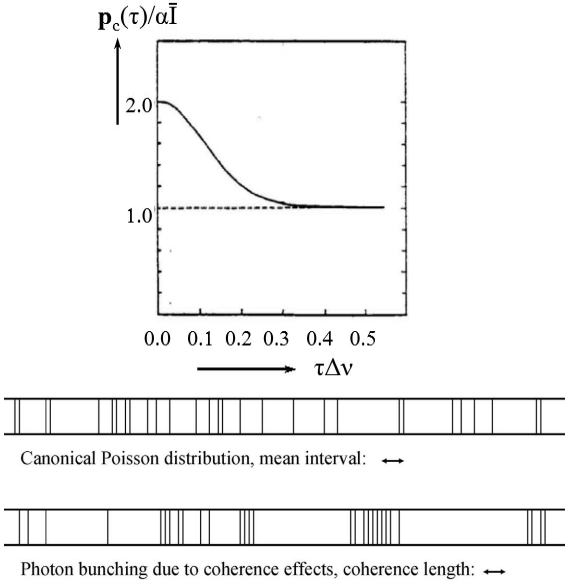

Figure 6.3: *Illustration of the 'bunching' effect of equation (6.9) (top) with an example of actual photon arrival times (bottom).*

optical correlation interferometer used by Brown and Twiss to measure stellar diameters.

6.3 Photon bunching

When a photon detector illuminated by 'Gaussian light' shows fluctuations in excess of Poisson noise, this should imply that the photon arrival does not occur fully at random but does exhibit a certain clustering or 'bunching' that is characteristic for bosons.
Consider a photon detector illuminated by polarized light with instantaneous intensity $I(t)$. Furthermore consider two small time intervals: from t to $t+dt$ and from $t+\tau$ to $t+\tau+d\tau$. The probability of observing

a photo-electric count at both times t and $t+\tau$ is then expressed as $\alpha^2 I(t)I(t+\tau)dtd\tau$, and the ensemble average $\alpha^2\overline{I(t)\cdot I(t+\tau)}dtd\tau$ gives the probability of finding two counts separated by an interval τ. Normalizing this to the probability of one count at t within dt we arrive at the conditional probability $\mathbf{p}_c(\tau)d\tau$ of obtaining a second count τ seconds after the first, i.e.:

$$\mathbf{p}_c(\tau)d\tau = \frac{\alpha^2\overline{I(t)\cdot I(t+\tau)}dtd\tau}{\alpha\bar{I}(t)dt} = \alpha\bar{I}\left[1+|\gamma_{11}(\tau)|^2\right]d\tau \qquad (6.15)$$

(see equation(6.9)).
Since $|\gamma_{11}(\tau)| \simeq 1$ for $\tau \ll 1/\Delta\nu$ and $|\gamma_{11}(\tau)| \ll 1$ for $\tau \gg 1/\Delta\nu$, the conditional probability always starts at the value $2\alpha\bar{I}d\tau$ and declines to $\alpha\bar{I}d\tau$ for large τ. The chance of detecting a photon close to another photon is therefore twice as large as for classical particles. Figure (6.3) shows an example of this 'bunching effect' for a Gaussian spectral frequency profile ($\tau_c \simeq (2\Delta\nu)^{-1}$) together with a counting distribution with degeneracy parameter $\delta \simeq 1$. The clustering is quite apparent. These results do not hold for maser or laser light in which intensity saturation effects occur. In this case the fluctuation will appear to be intermediate between the illustrated examples.

6.4 Stellar intensity correlation interferometry

R. Hanbury Brown and Richard C. Twiss have pioneered the use of correlation interferometry both in the radio and optical wavebands. A brief discussion of the principle of their stellar intensity correlation interferometer is given here, the measurement set up is shown in figure 6.4.

Consider the light beams incident on the two mirrors. One can now write for the cross-correlation of the intensity fluctuations ΔI_1 and ΔI_2:

$$\overline{\Delta I_1(t)\cdot\Delta I_2(t+\tau)} = \frac{1}{2}\bar{I}_1\bar{I}_2|\gamma_{12}(\tau)|^2 \qquad (6.16)$$

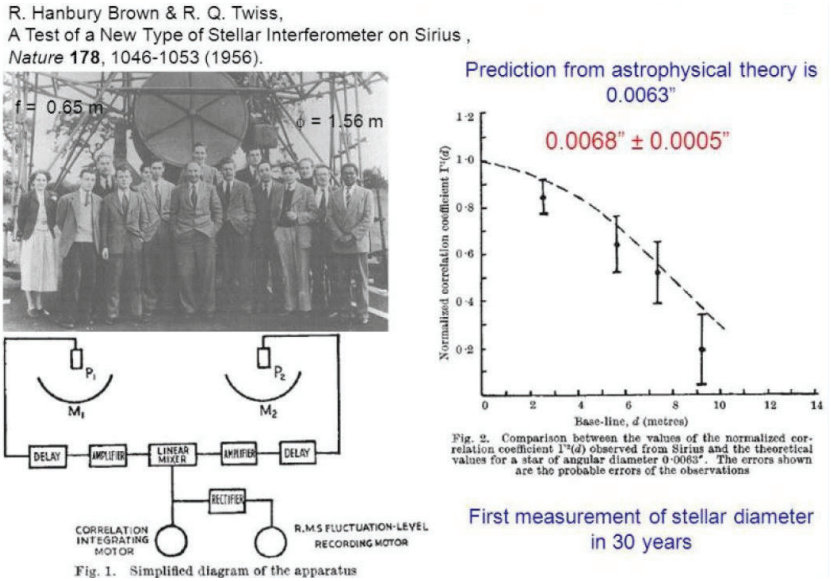

Figure 6.4: *The WWII search lights in Manchester with which Hanbury Brown and Twiss measured the diameter of Sirius in 1956 and the correlation as a function of baseline obtained in this measurement (right). Credit Hanbury Brown and Twiss (1956).*

This formula is analogous to the expression of the autocorrelation for which positions 1 and 2 coincide, $\gamma_{12}(\tau)$ is the complex degree of coherence, the factor $\frac{1}{2}$ arises from the fact that the radiation beam is unpolarised. For $\tau = 0$ one has $\overline{\Delta I_1(t) \cdot \Delta I_2(t)} = \frac{1}{2}\bar{I}_1\bar{I}_2|\gamma_{12}(0)|^2$, with $|\gamma_{12}(0)| \equiv V$, the visibility function that can be employed to assess the angular diameter θ_s of a star. Next, consider the stellar disk as a circular distribution of incoherent point sources with uniform brightness, so that its visibility fringes are described by equation (5.41) with the

substitution $\rho = h$:

$$|\gamma_{12}(0)| = 2\left|\frac{J_1(\pi h\theta_s/\bar{\lambda})}{\pi h\theta_s/\bar{\lambda}}\right| \tag{6.17}$$

Since the resolving time T of the measuring equipment substantially exceeds the coherence time of the light beams, it reduces the coherence effect by τ_c/T as explained in the previous sections. In most case the assumption $\gamma_{12}(\tau) = \gamma_{12}(0)\gamma_{11}(\tau)$ can be made, hence:

$$\overline{\Delta I_1(t)_T \cdot \Delta I_2(t)_T} = \frac{1}{2}\bar{I}_1\bar{I}_2\frac{\tau_c}{T}|\gamma_{12}(0)|^2 \tag{6.18}$$

With their stellar intensity correlation interferometer Brown and Twiss measured this product (the correlator is a multiplier) at various separations h by changing the distance between the two telescopes, thereby tracking the function $|\gamma_{12}(0)|^2$ as a function of h. The angular diameter θ_s is then obtained by fitting the theoretical coherence pattern of equation (6.17), (6.18) to the measurements. The first star for which a diameter was determined this way was Sirius, for which Brown and Twiss found $\theta_s = 0.0068'' \pm 0.0005''$.

The big potential of the intensity interferometer above the Michelson interferometer is that the separation h can be made much larger because one does not have to worry about spoiling the phase of the lightwaves: in a Michelson interferometer the distance between the two mirrors must be kept constant to within a fraction of a wavelength, whereas the intensity interferometer ignores phase effects. An added bonus is that the quality of the mirrors is not critical, Brown and Twiss used in fact two World-War II search-light mirrors for their first experiments. A list of angular diameters for stars measured by Brown with the intensity interferometer at Narrabri is given in table 6.1.

The main limiting factor for the intensity interferometer is the very high photon arrival probability density that is required in order to obtain a non-negligible value of the degeneracy parameter $\delta = \bar{n}_o\tau_c$ ($\bar{n_o}$ is the average photo-electron rate). Consequently, only stars with magnitude $V \leq 3$ can be measured. This constraint gets worse at shorter wavelengths and the method cannot be used in the ultraviolet or X-rays.

Table 6.1: *List of stars measured at the Narrabri Observatory*

star	Spectral type	angular diameter (uniform disk) in milliarcsec	angular diameter (limb-darkened disk) in milliarcsec
β Cru	B0.5 IV	0.705±0.025	0.728±0.026
γ Ori	B2 III	0.74±0.05	0.76±0.05
ϵ CMa	B2 II	0.78±0.05	0.81±0.05
α Pav	B3 IV	0.77±0.06	0.80±0.06
ϵ Ori	B0 Ia	0.70±0.05	0.72±0.05
α Eri	B5 IV	1.86±0.07	1.93±0.08
α Gru	B5 V	0.98±0.07	1.02±0.07
α Leo	B7 V	1.33±0.07	1.38±0.07
β Ori	B8 Ia	2.57±0.14	2.69±0.15
α CMa	A1 V	5.85±0.10	6.12±0.10
α Lyr	A0 V	3.31±0.15	3.47±0.16
α PsA	A3 V	1.88±0.13	2.09±0.14
α Car	F0 Ib-II	6.48±0.39	6.86±0.41
α Aql	A7 IV-V	2.79±0.14	2.97±0.15
α CMi	F5 IV-V	5.31±0.36	5.71±0.39

The radio range is the best suitable regime, and the first results were obtained in that band – but not of course for thermal sources.

Chapter 7

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